Image processing 101

Fourier Fun

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Images in computers

- Discretized in space $\Longrightarrow$ Pixels *picture elements*
- Discretized in intensitiy/color
  - Resolution: 8-bit, 14-bit, 16-bit . . .
  - Color representation: RGB (additive: red green blue), CMY (subtractive: cyan magenta yellow), HSV (hue saturation brightness; color wheel)
  - Color: quantized palette (8-bit) or true color (24-bit)
- Registration:
  - Photographic film + photometer
  - vacuum tube (TV camera)
  - solid-state devices (CCD, CID)
- Transmission: Serialization
  - Bandwidth limitations

Linear Bases

Linear decomposition of functions:

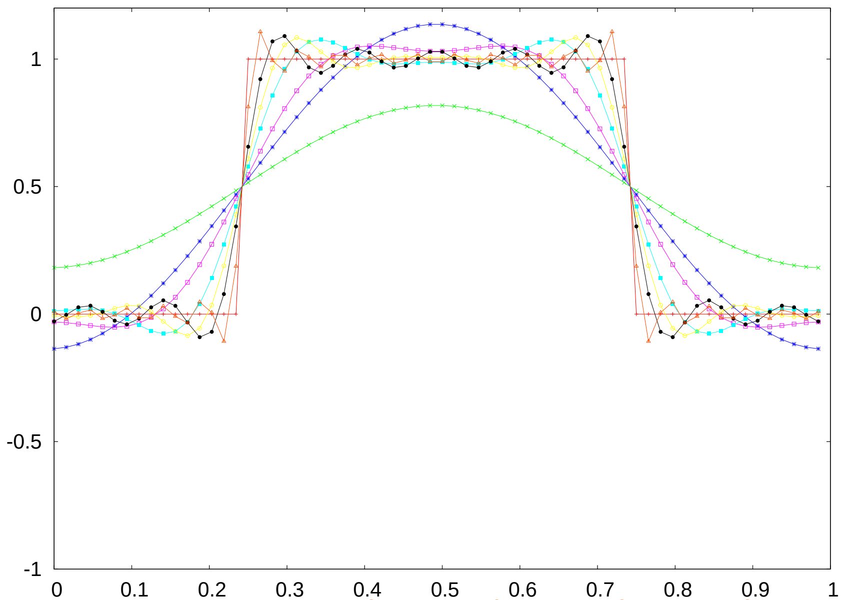
Fourier coefficient

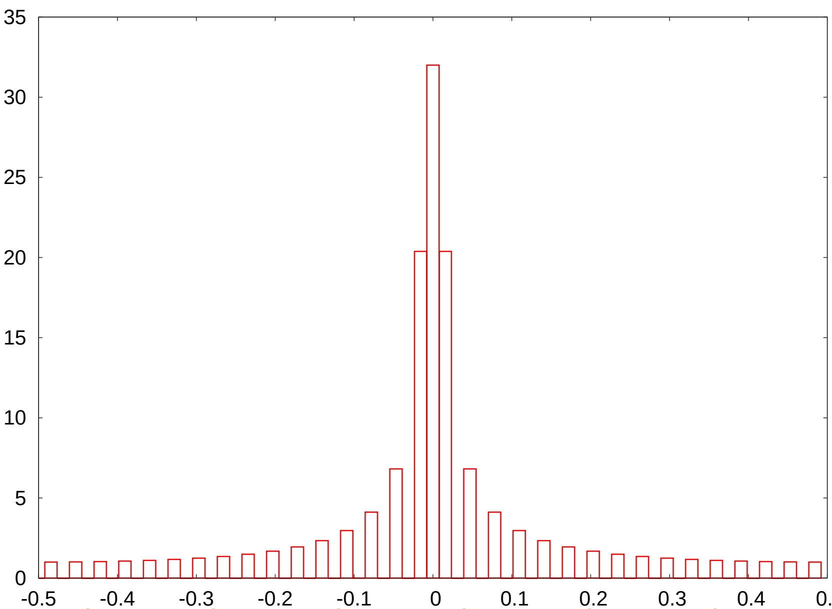
Fourier basis function

$$f(x) = \sum_k c_k \, b_k(x)$$

Fourier series expansion:

$$b_k(x) = e^{ik \cdot x}$$





Plane waves and complex numbers

Complex numbers

norm phase

$$C = \mathrm{Re}\, C + i\ \mathrm{Im}\, C = |C|\, e^{i\phi} = |C|\, (\sin\phi + i\ \cos\phi)$$

$$C^* = \mathrm{Re}\, C - i\ \mathrm{Im}\, C$$

$$|C| = \sqrt{C^* C} = \sqrt{(\mathrm{Re}\, C)^2 + (\mathrm{Im}\, C)^2}$$

Plane waves

$$b_k(x) = e^{ik\cdot x} = \sin k\cdot x + i\ \cos k\cdot x$$

$$|b_k(x)| \equiv 1$$

Interference

$$b_{k_1}(x) + b_{k_2}(x) = e^{ik_1\cdot x} + e^{ik_2\cdot x}$$

$$|b_{k_1}(x) + b_{k_2}(x)|^2 = 2 + 2\cos[(k_1 - k_2)\cdot x]$$

The Fourier transform

Fourier synthesis:

$$f(x) \quad = \quad \frac{1}{N} \sum_k \hat{f}(k)\, e^{ik \cdot x}$$

Fourier analysis:

$$\hat{f}(x) = (\mathcal{F} f)(x) \quad = \quad \sum_x f(x)\, e^{-ik \cdot x}$$

$$= \quad \langle f, b_k^* \rangle$$

Scalar product

k is the *wavenumber*: Frequency ν

$$\nu = \frac{k}{2\pi}$$

Linearity:

$$\mathcal{F}(f + g) \quad = \quad \mathcal{F} f + \mathcal{F} g$$

$$\mathcal{F}(\alpha f) \quad = \quad \alpha\, \mathcal{F} f$$

The Fourier transform

- Two different, complete representation of the same signal
  - Position/time space
    Localization of signals
  - Momentum/frequency space
    Repetition/Periodicity and shape
    Translation-invariant basis
- Power spectrum

$$\sum_x |f(x)|^2 = \sum_k |\hat{f}(k)|^2$$

- Fast-fourier transform
  - $O(N \log N)$

Fourier modes and reciprocity

Discretization:

$$x = 0, \Delta x, 2\Delta x, \ldots, L - 2\Delta x, L - \Delta x$$

Periodicity:

$$f(x) = f(x + L)$$

For plane waves:

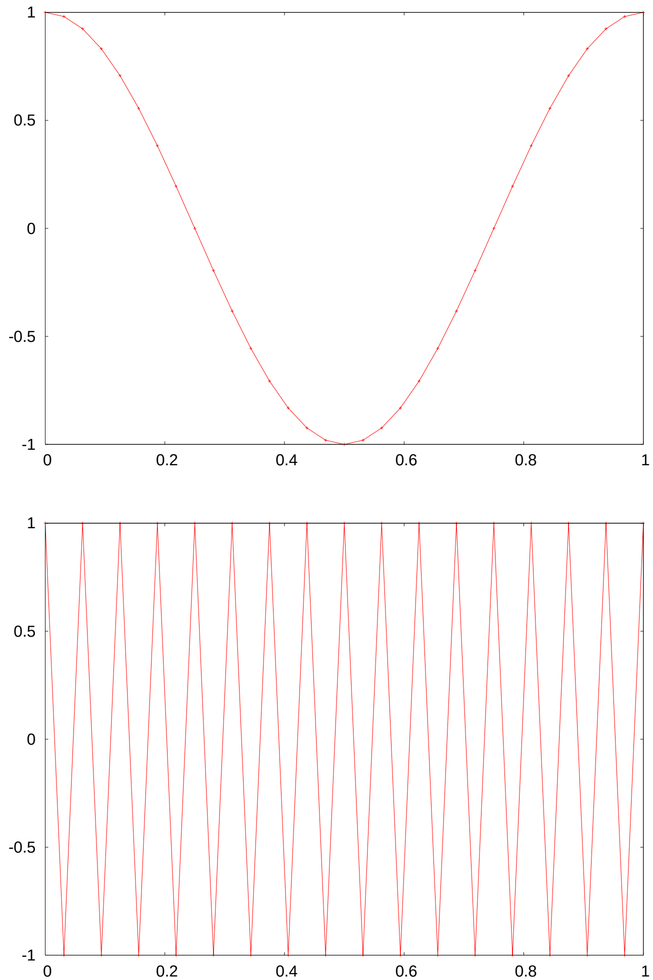
$$e^{ikx} = e^{ik(x+L)} = e^{ikx} e^{ikL} \qquad \Rightarrow \qquad e^{ikL} = 1$$

$$kL = 2\pi n \qquad \Rightarrow \qquad \boxed{k = \frac{2\pi}{L} n}$$

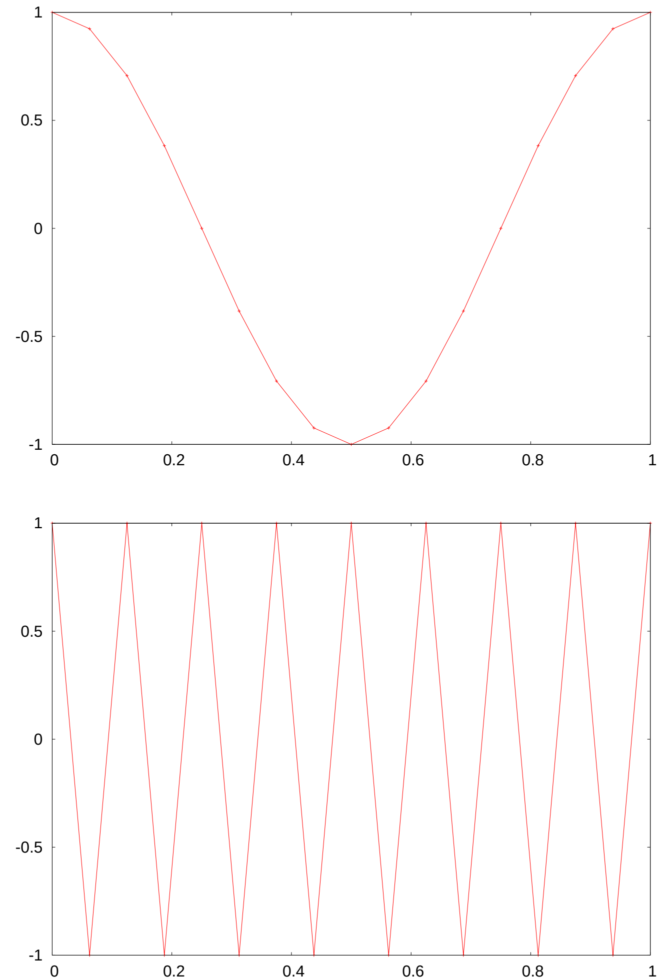
Highest-possible frequency: $(-1, +1, -1, +1, \ldots)$

$$(-1)^{x/\Delta x} = e^{-\pi(x/\Delta x)} \qquad \Rightarrow \qquad \boxed{k_{\mathrm{max}} = \frac{\pi}{\Delta x} = \frac{2\pi}{L} \frac{N}{2}} \qquad \nu_{\mathrm{max}} = \frac{1}{2\Delta x}$$

Fourier modes and reciprocity



$L = 1,\ \Delta x = \frac{1}{32}$

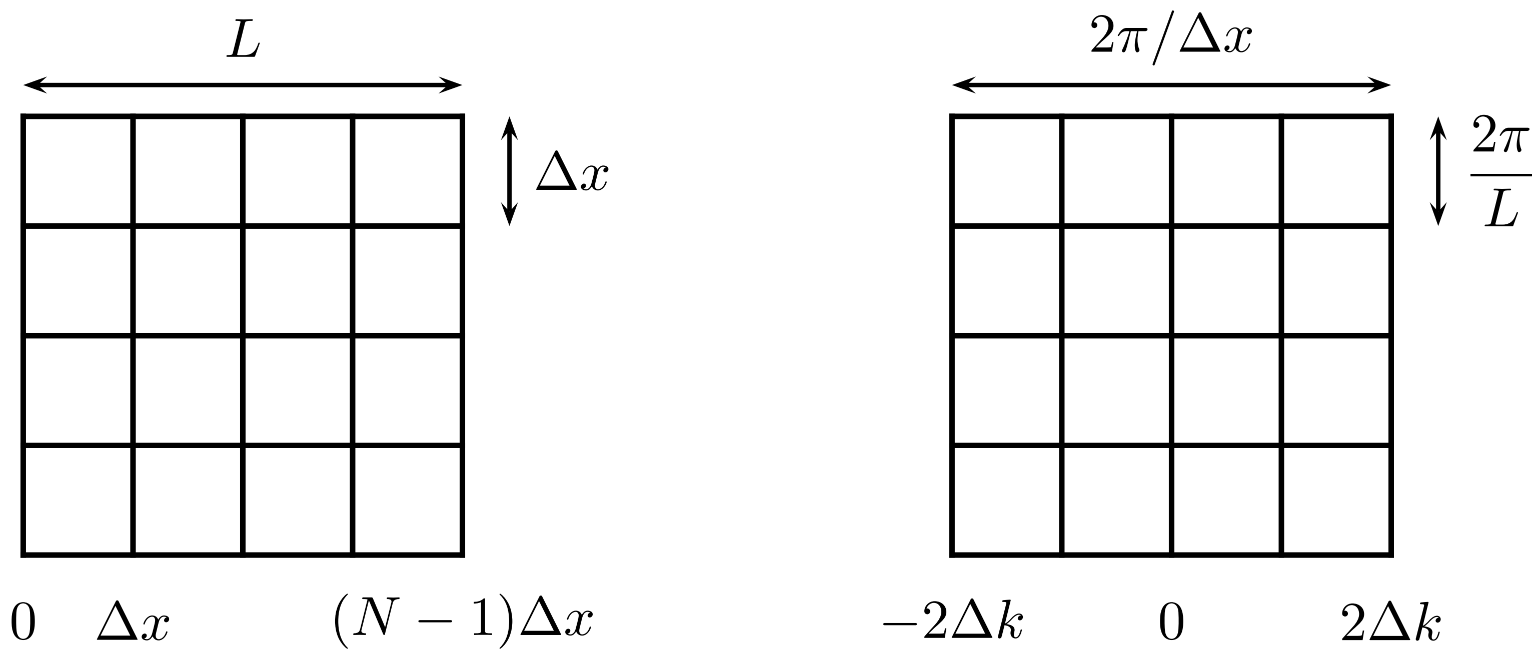


$L = 1,\ \Delta x = \frac{1}{16}$

Dual lattices

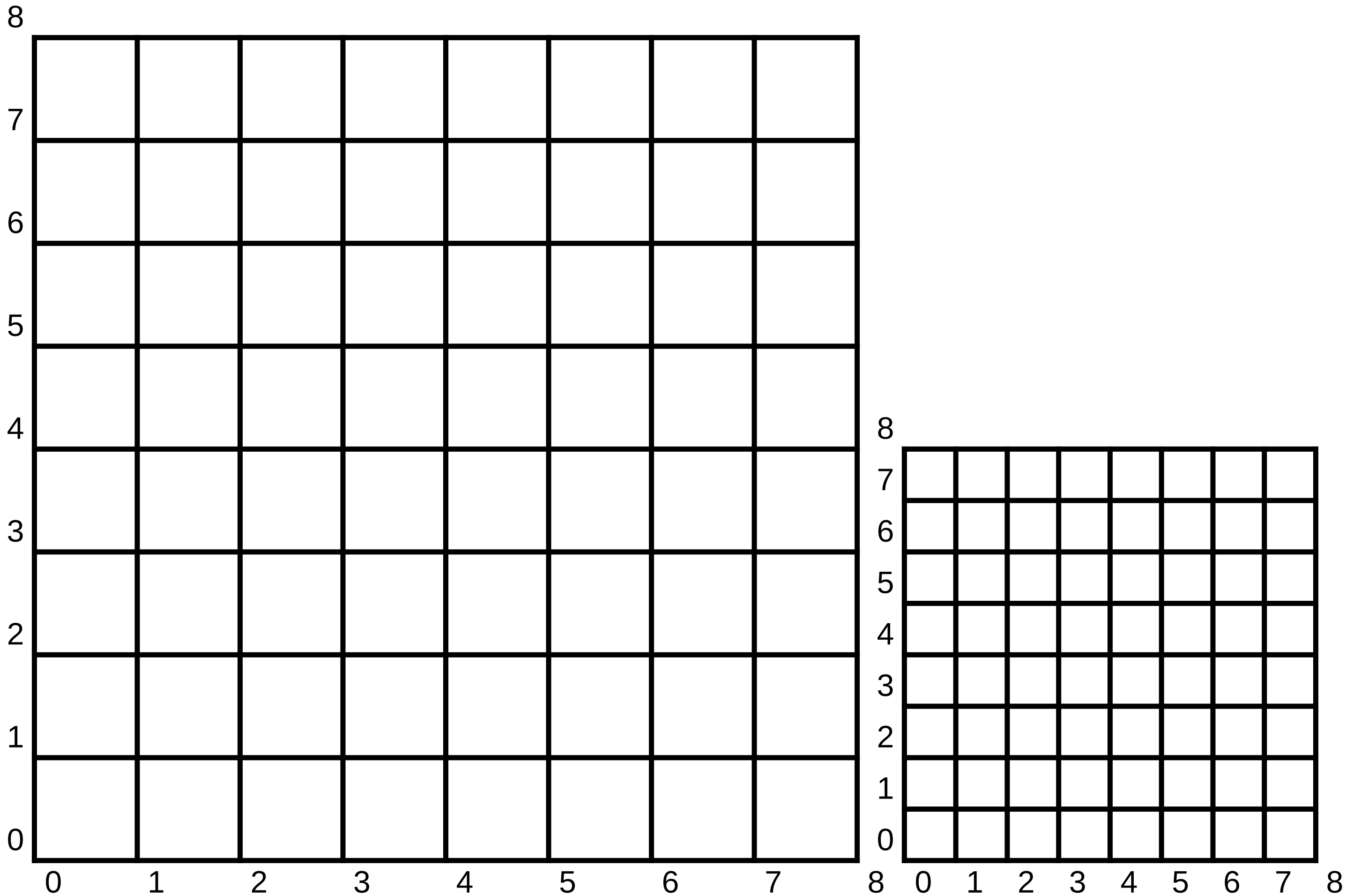
$$\Delta k = \frac{2\pi}{L}$$

$$x = 0, \Delta x, \ldots, L - \Delta x \qquad k = -\Delta k \frac{N}{2}, \ldots, \Delta k \left(\frac{N}{2} - 1 \right)$$



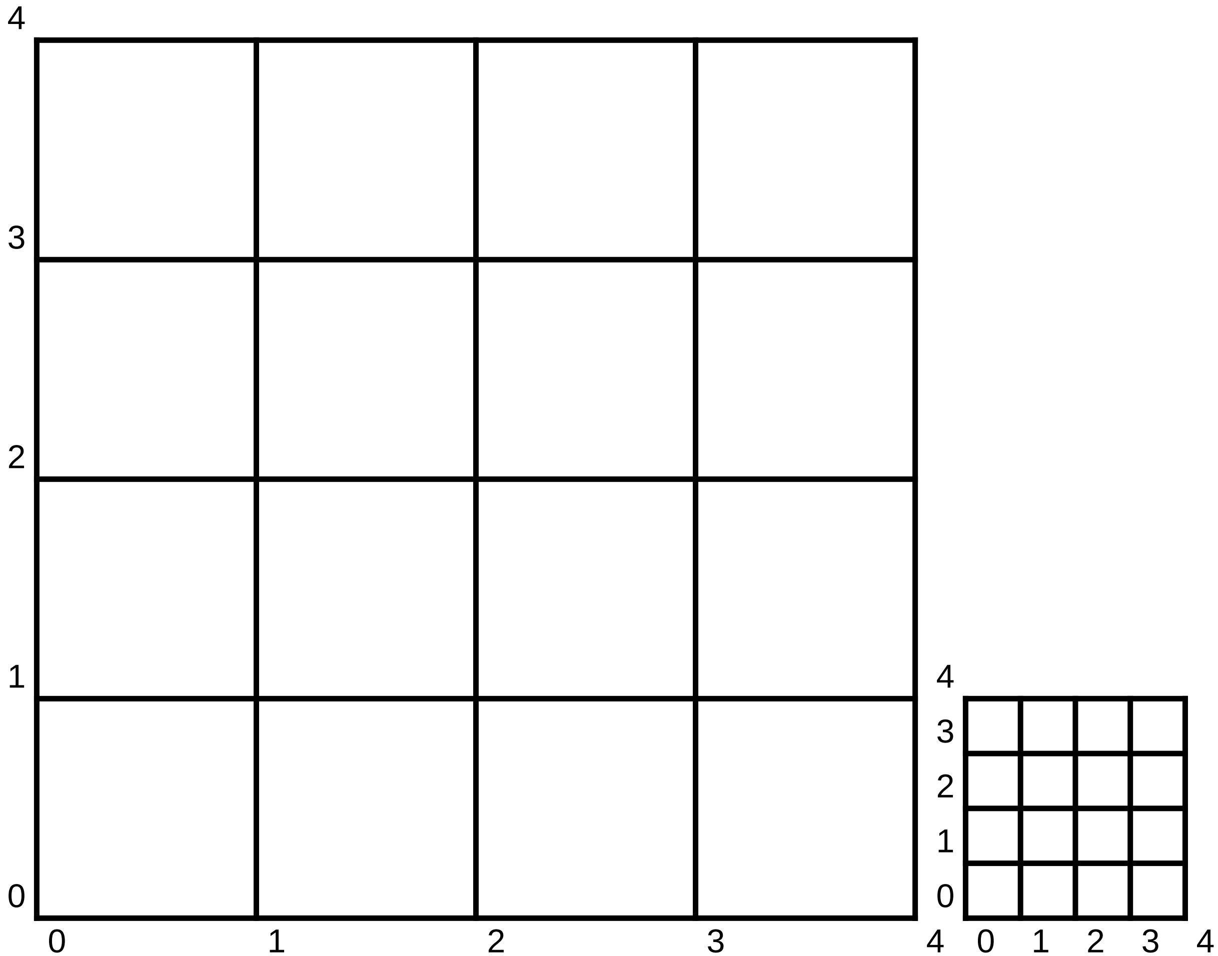
Dual lattices

$$L \longrightarrow 2L \qquad \Rightarrow \Delta k \longrightarrow \frac{\Delta k}{2}$$

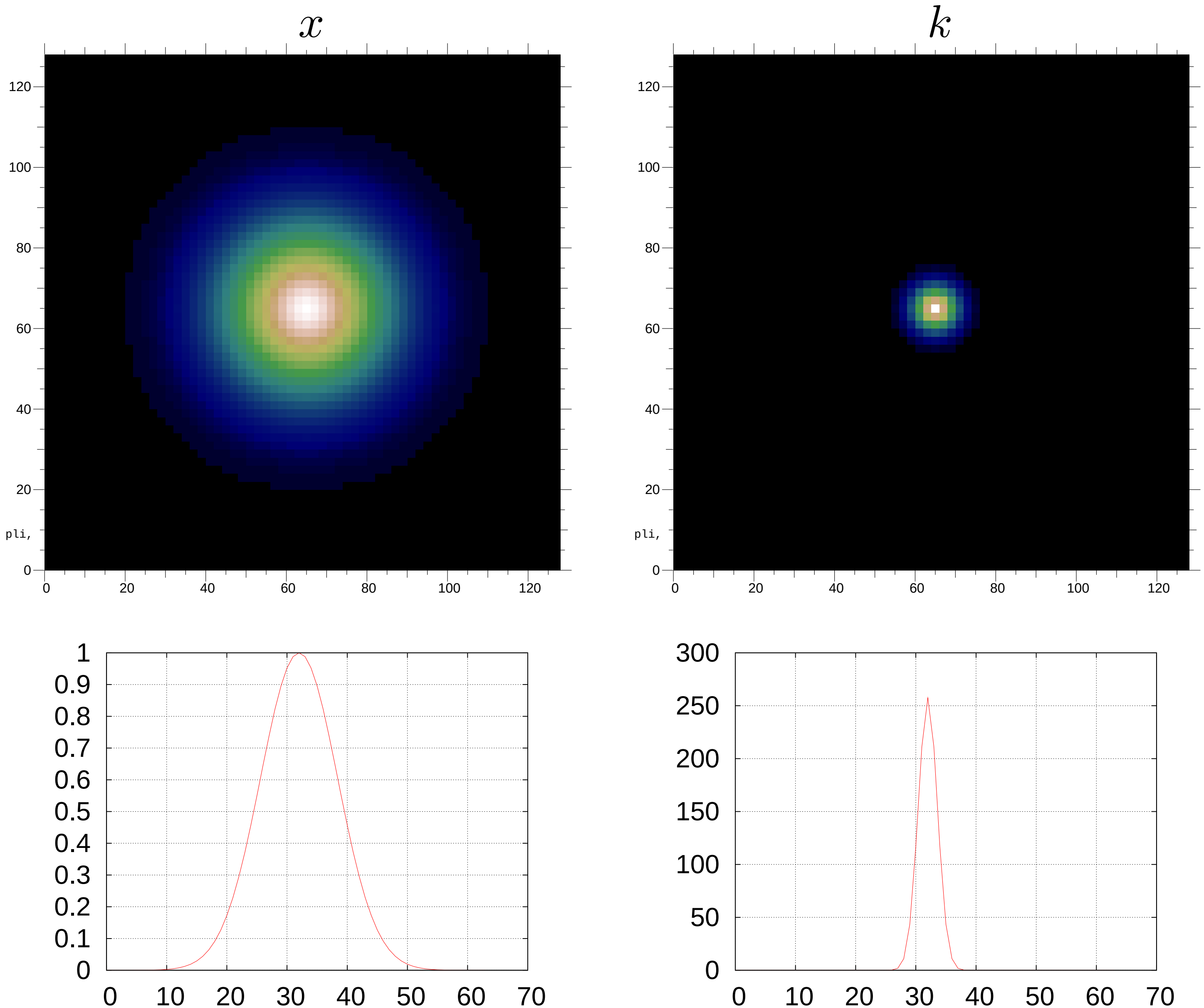


Dual lattices

$$\Delta x \longrightarrow 2\Delta x \qquad \Rightarrow k_{\max} \longrightarrow \frac{k_{\max}}{2}$$



Blob in Fourier space



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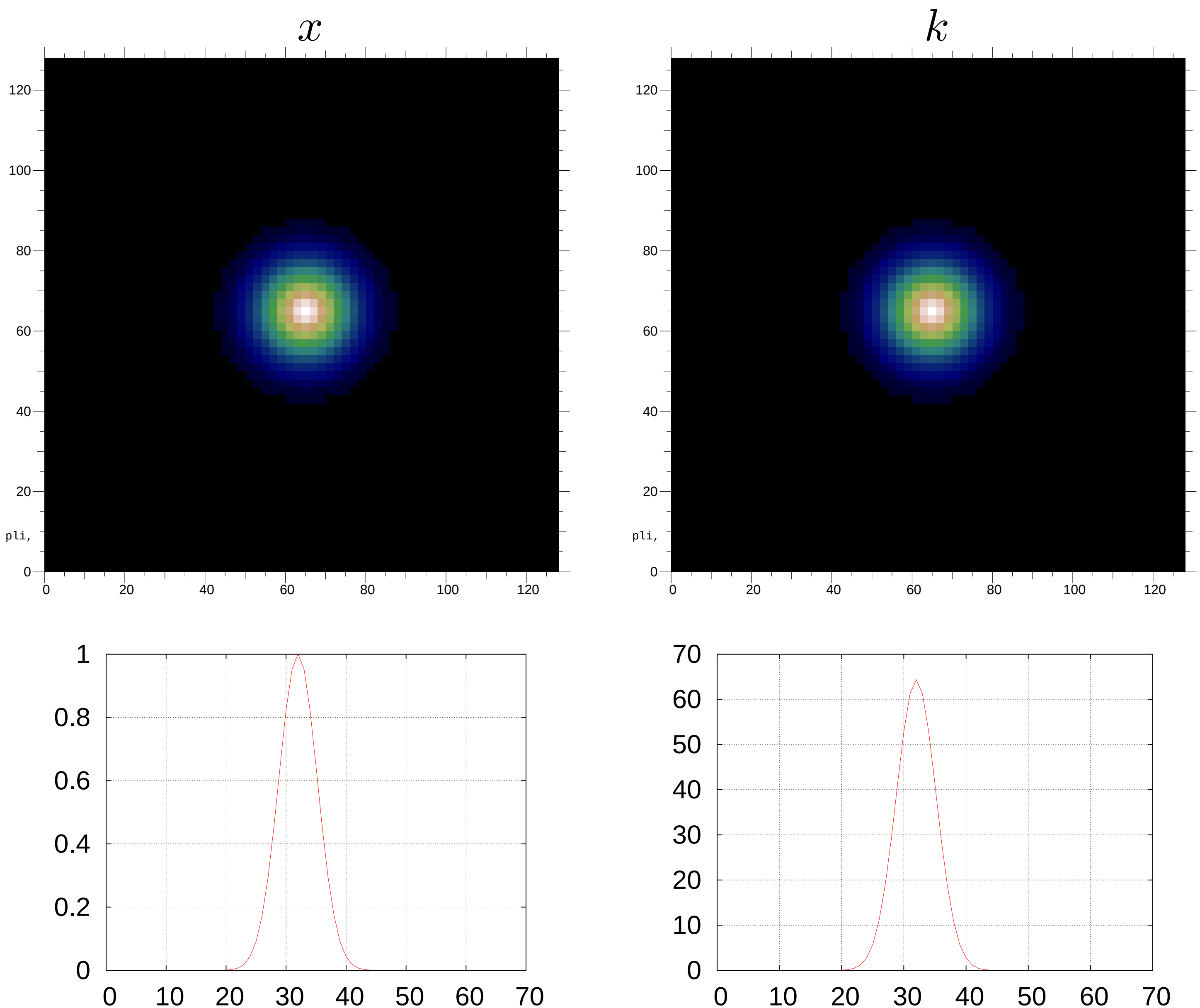

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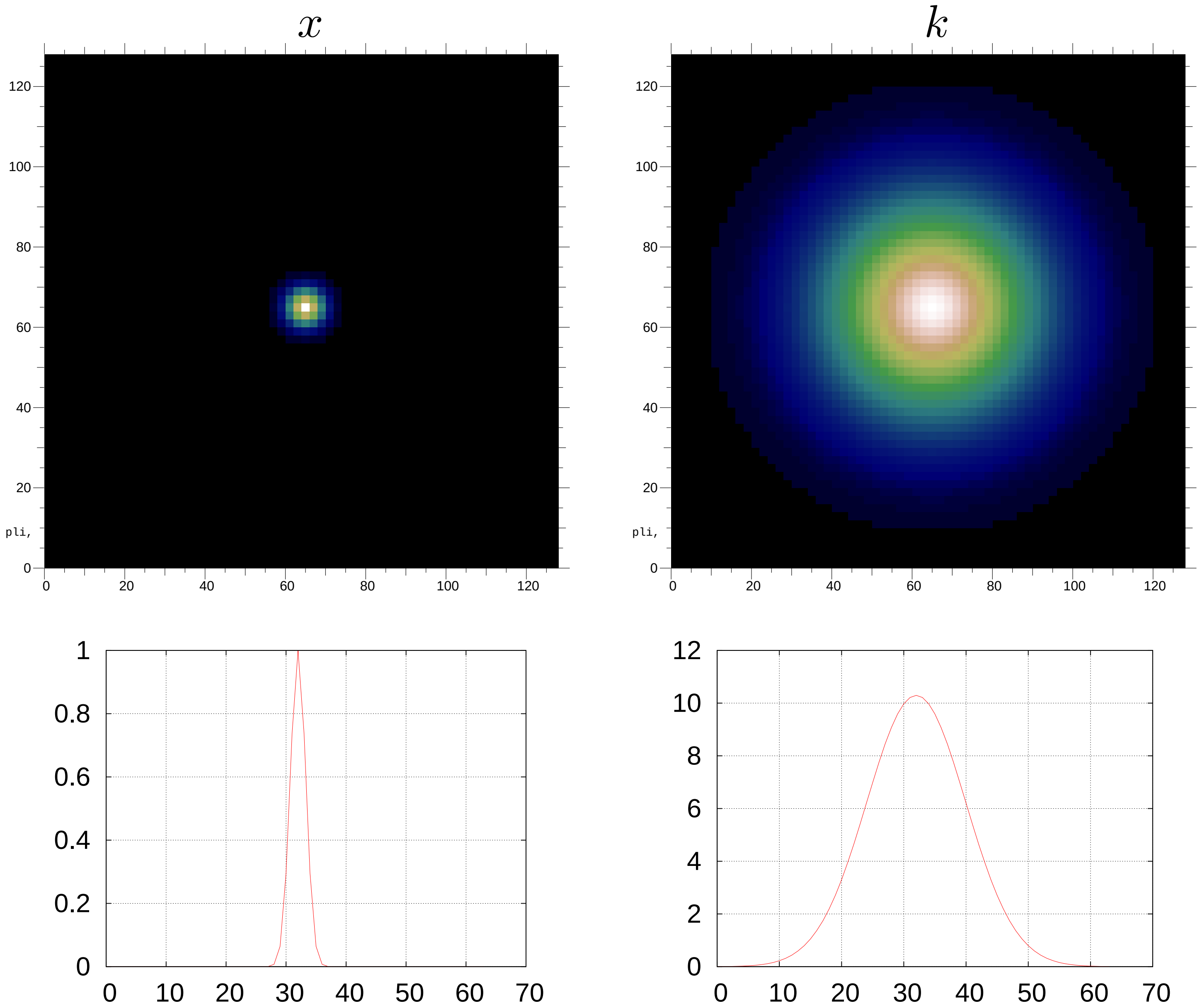

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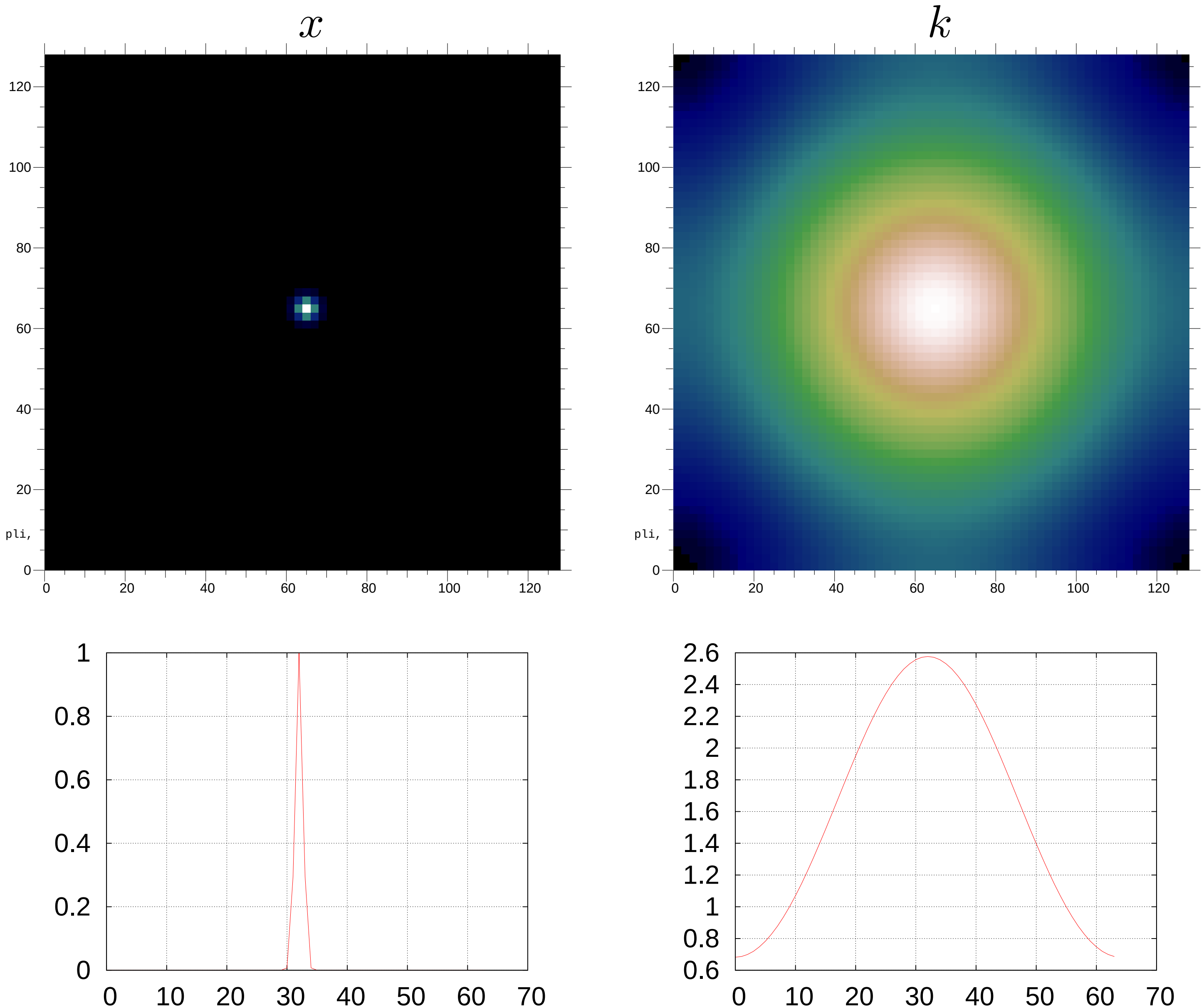

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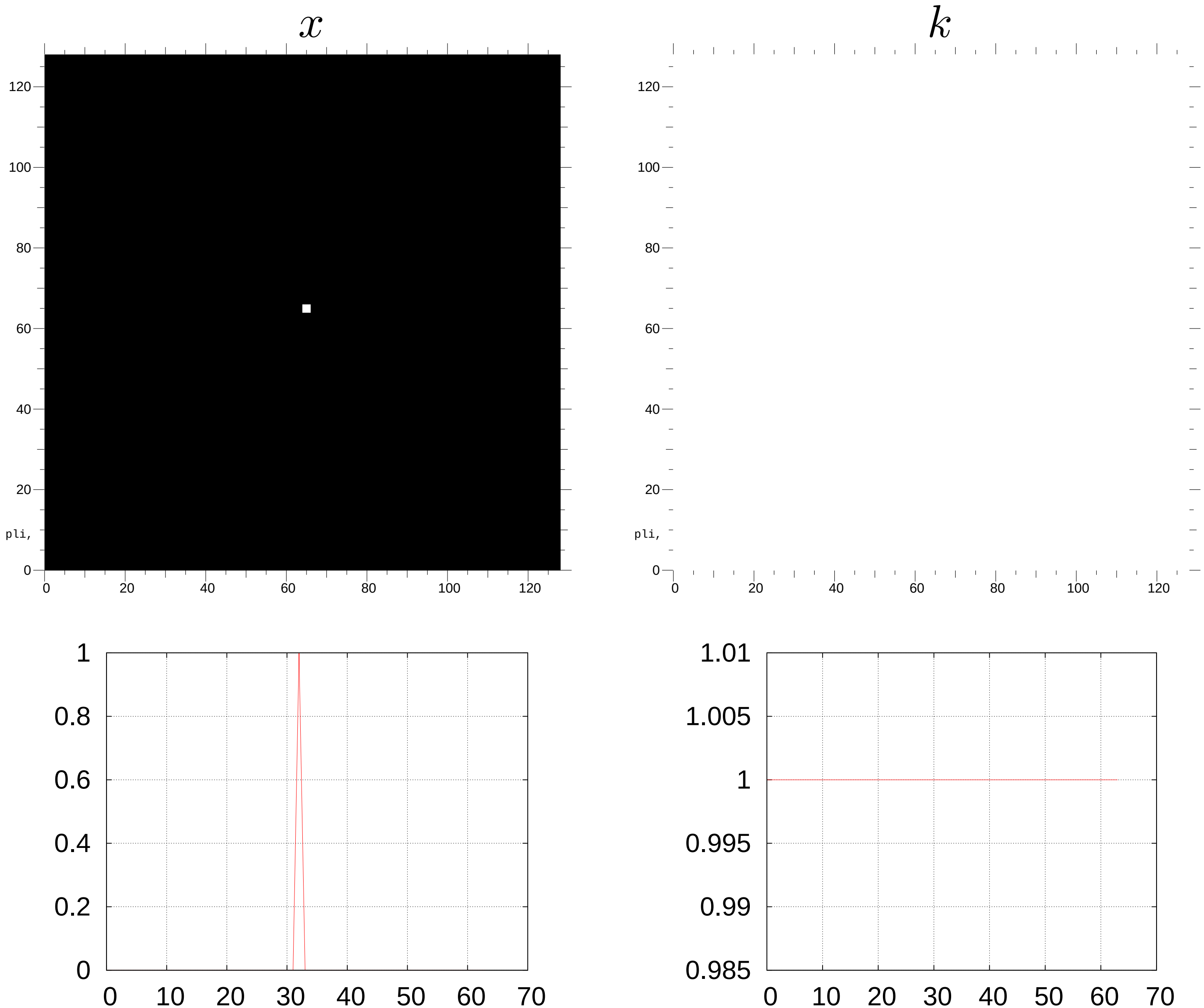

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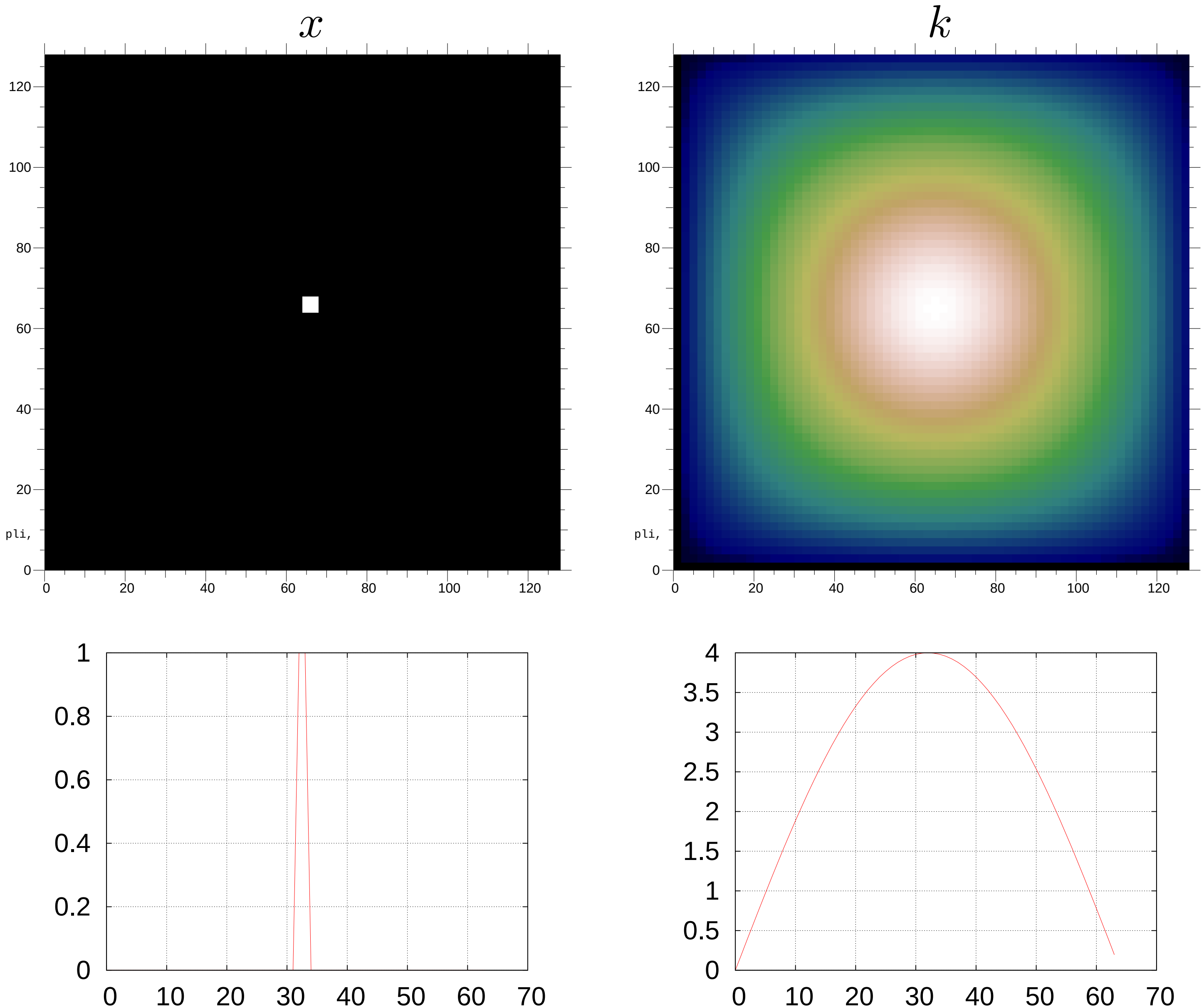

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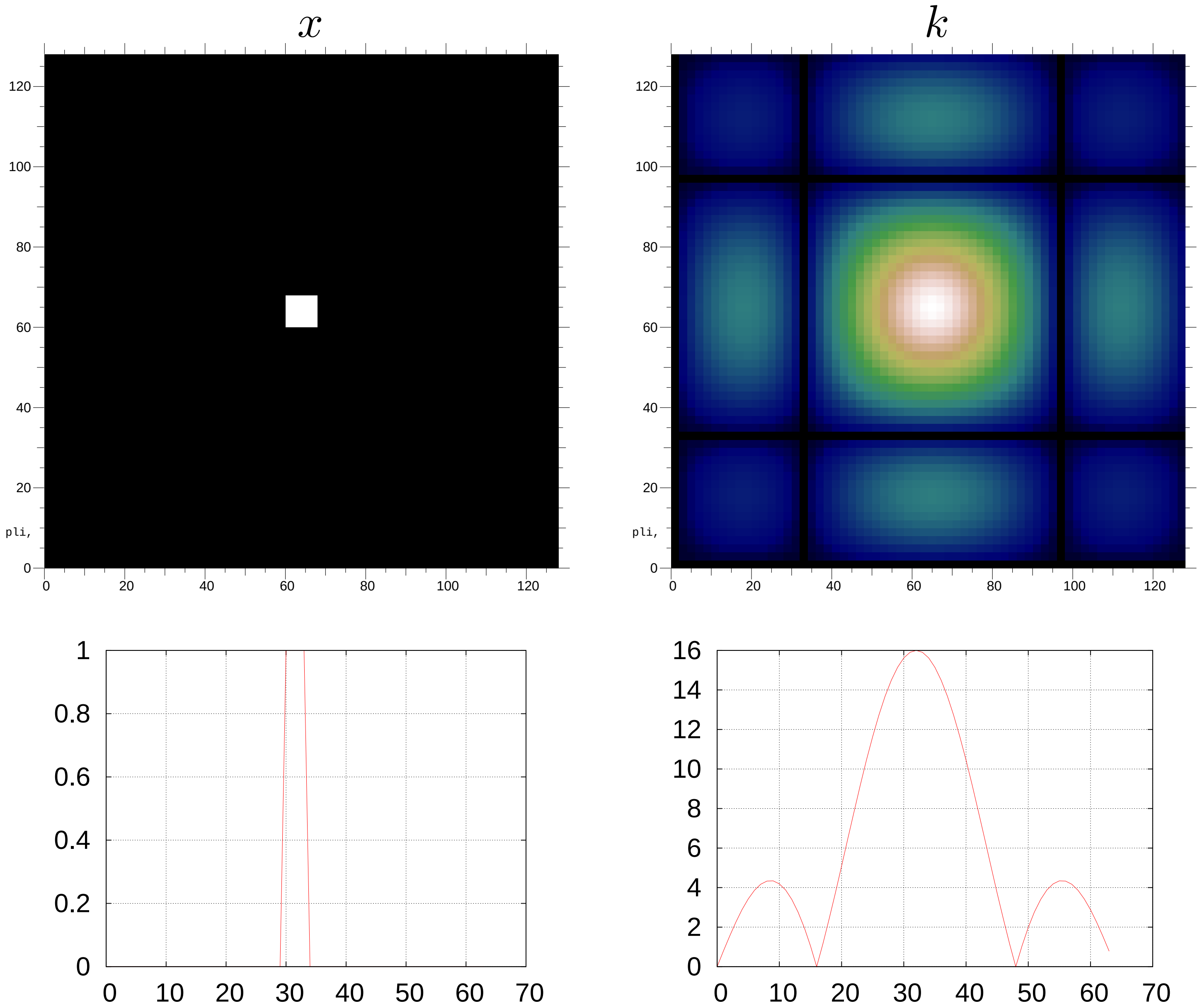

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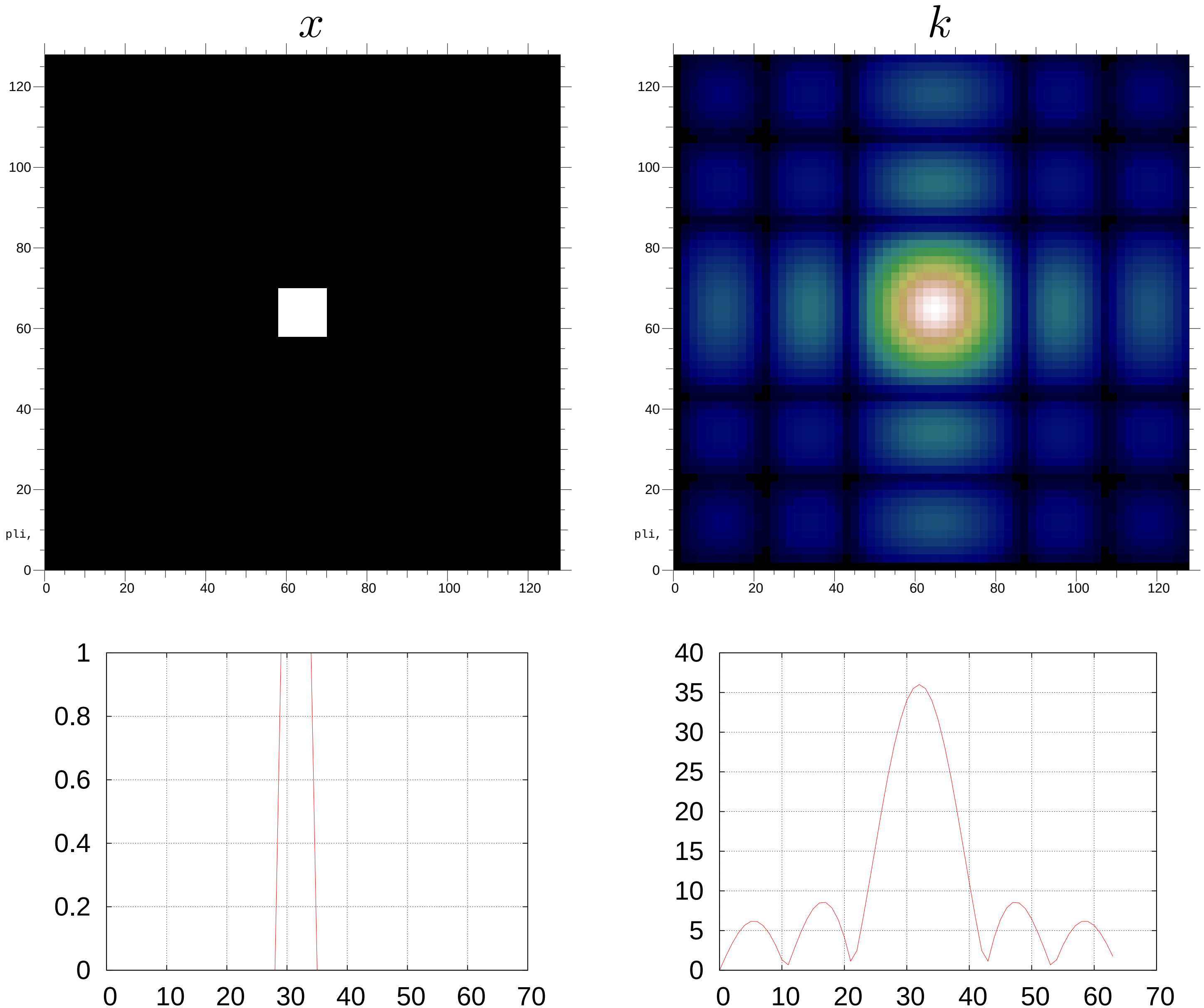

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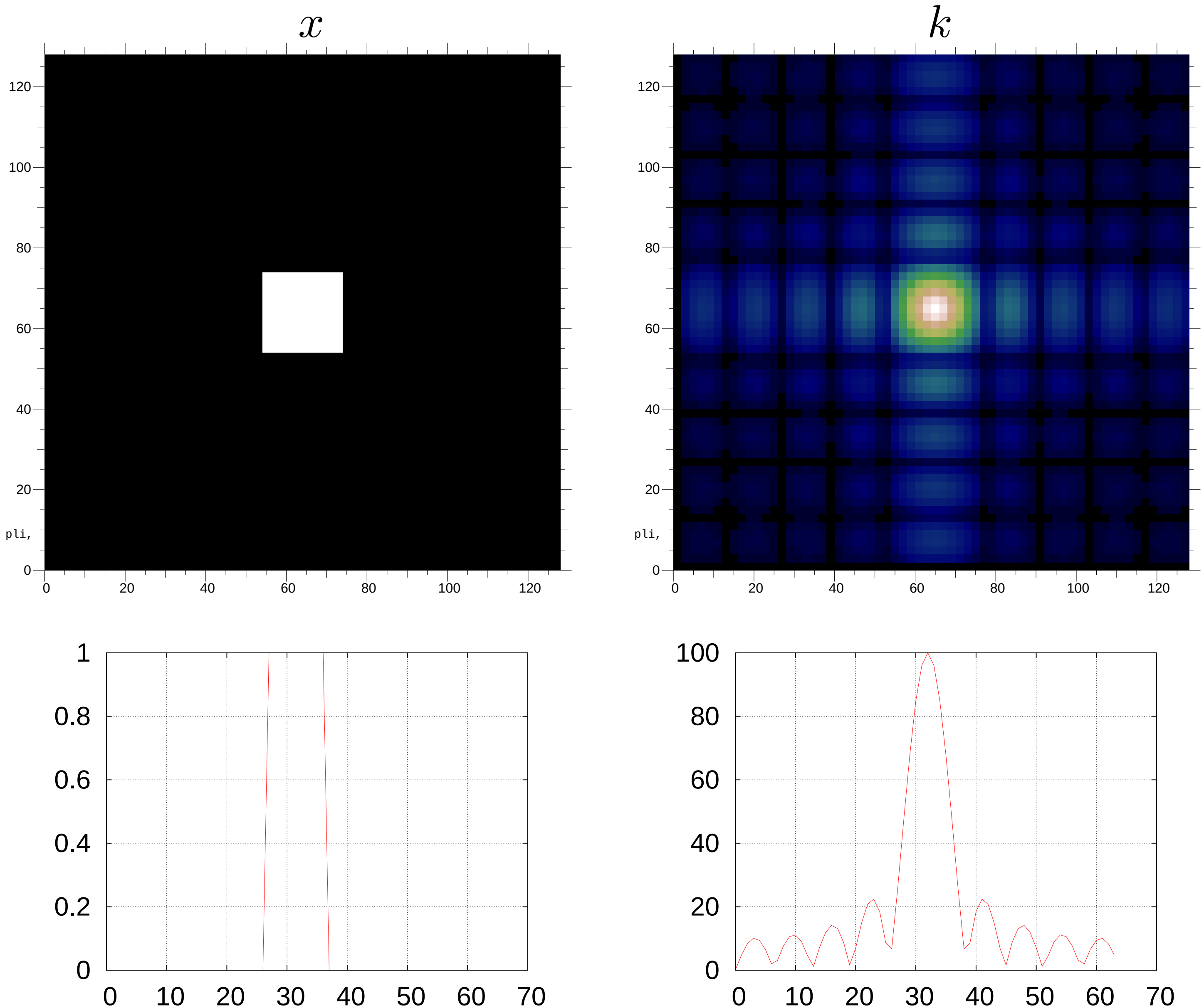

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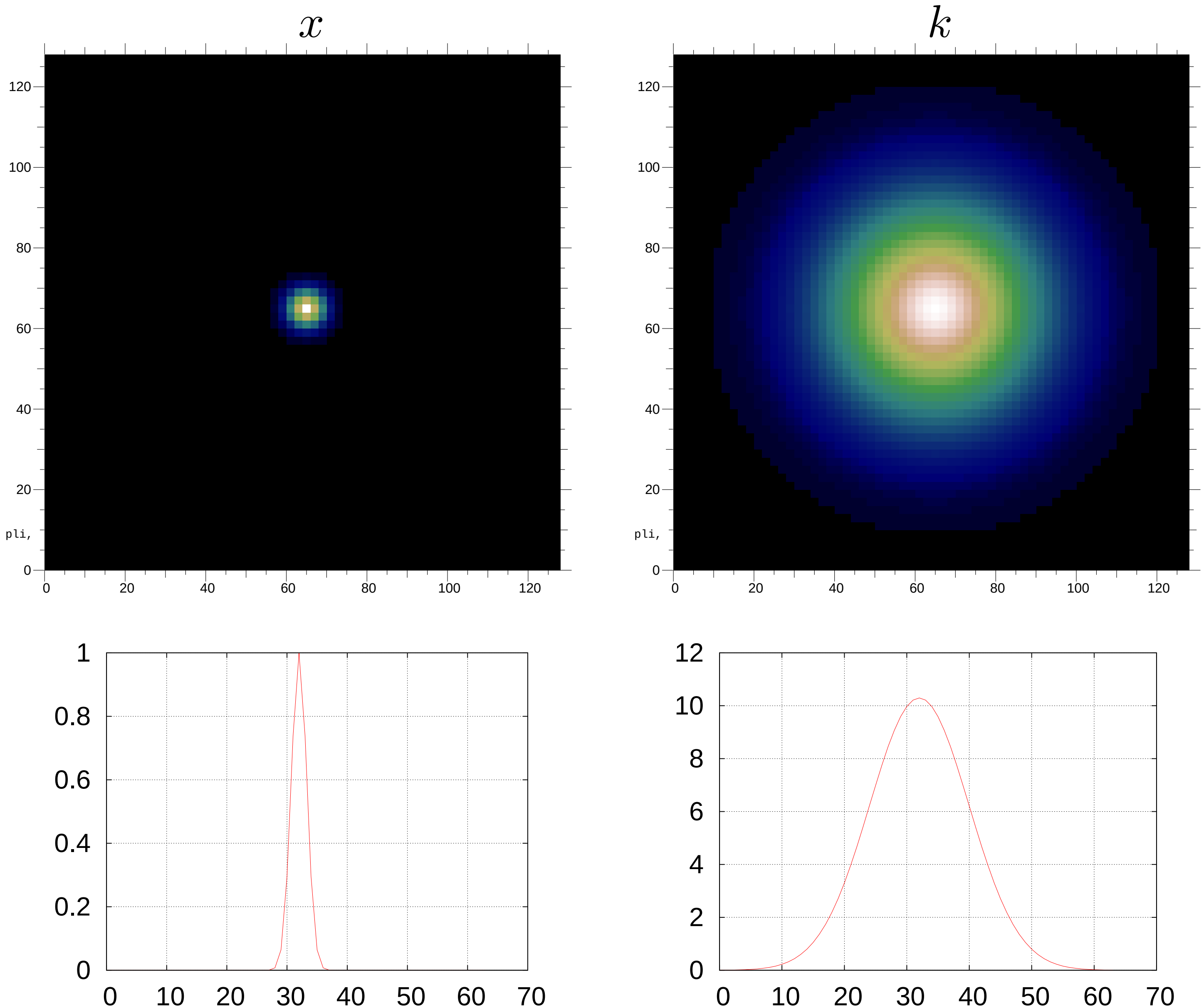

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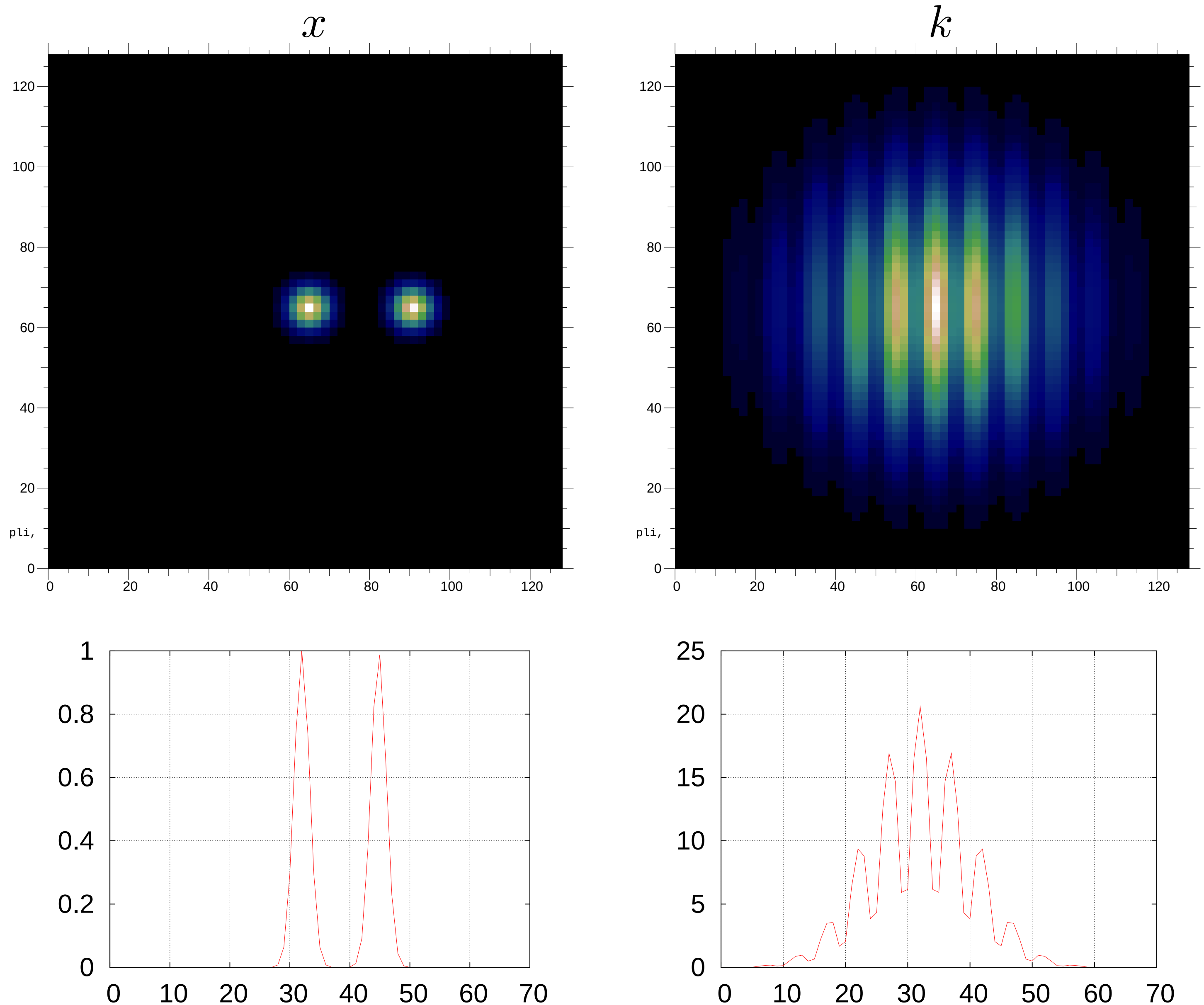

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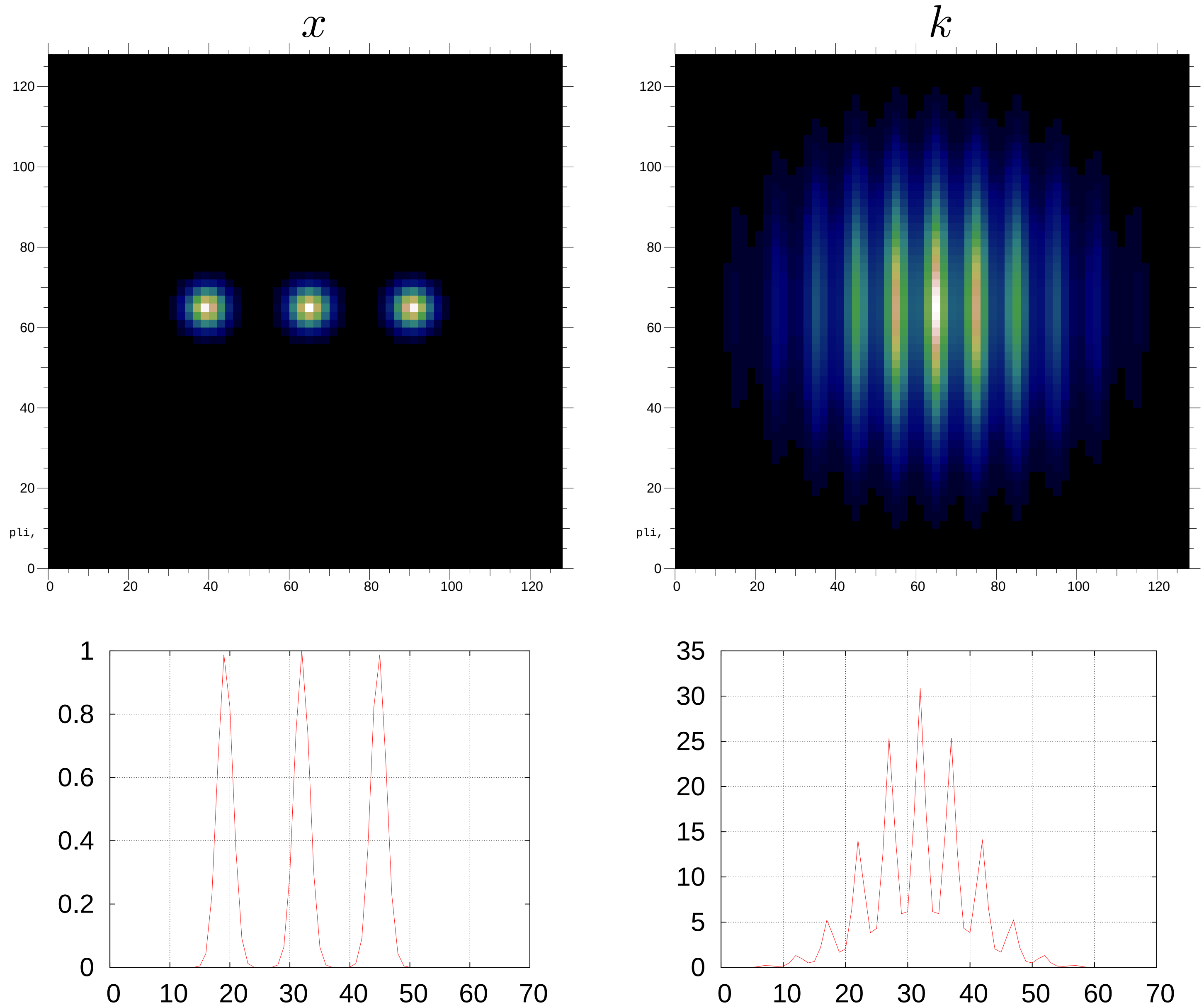

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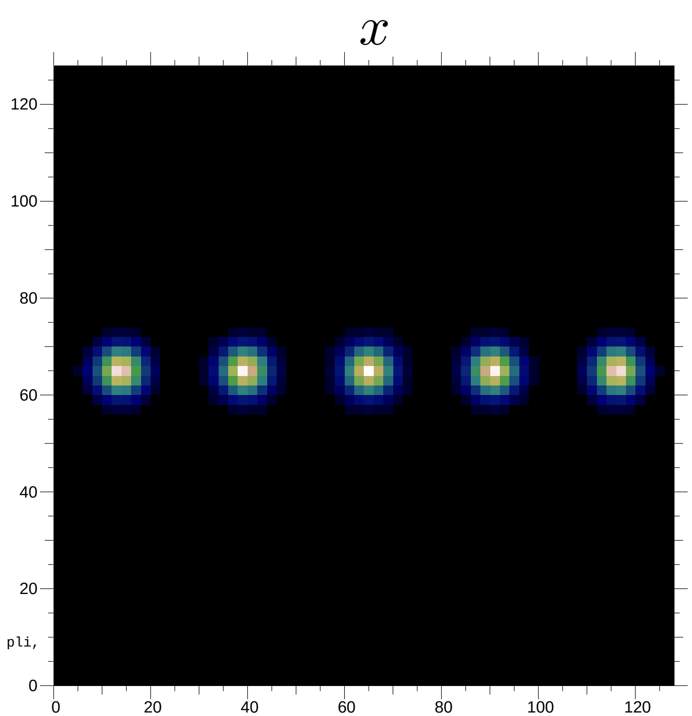
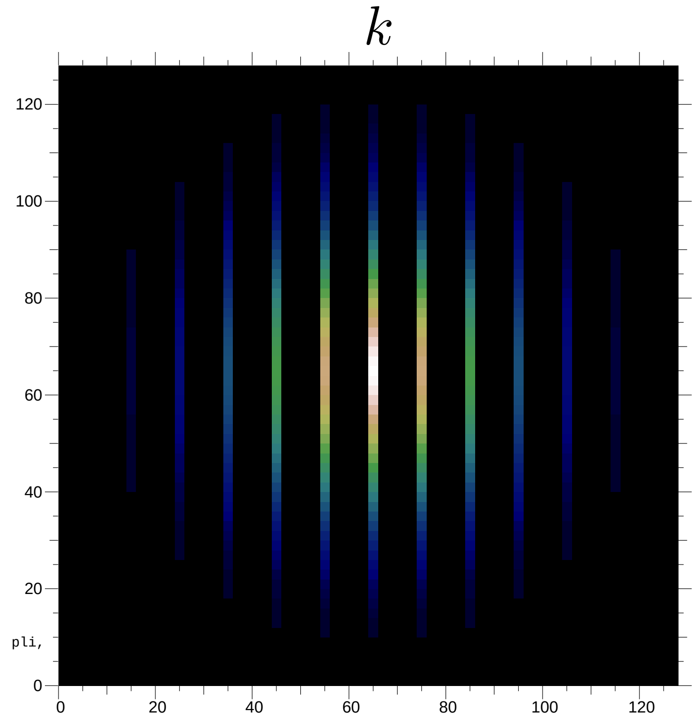
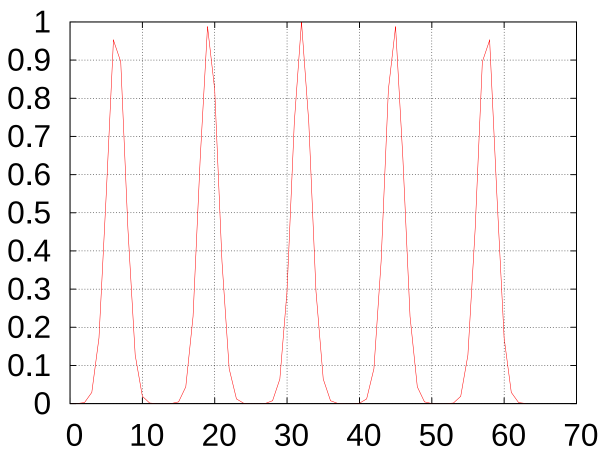
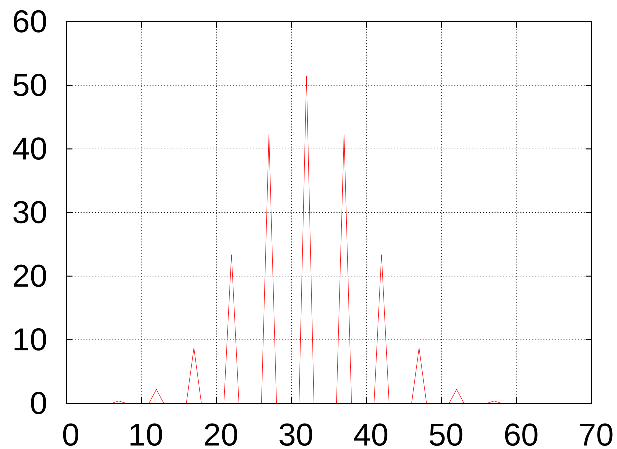
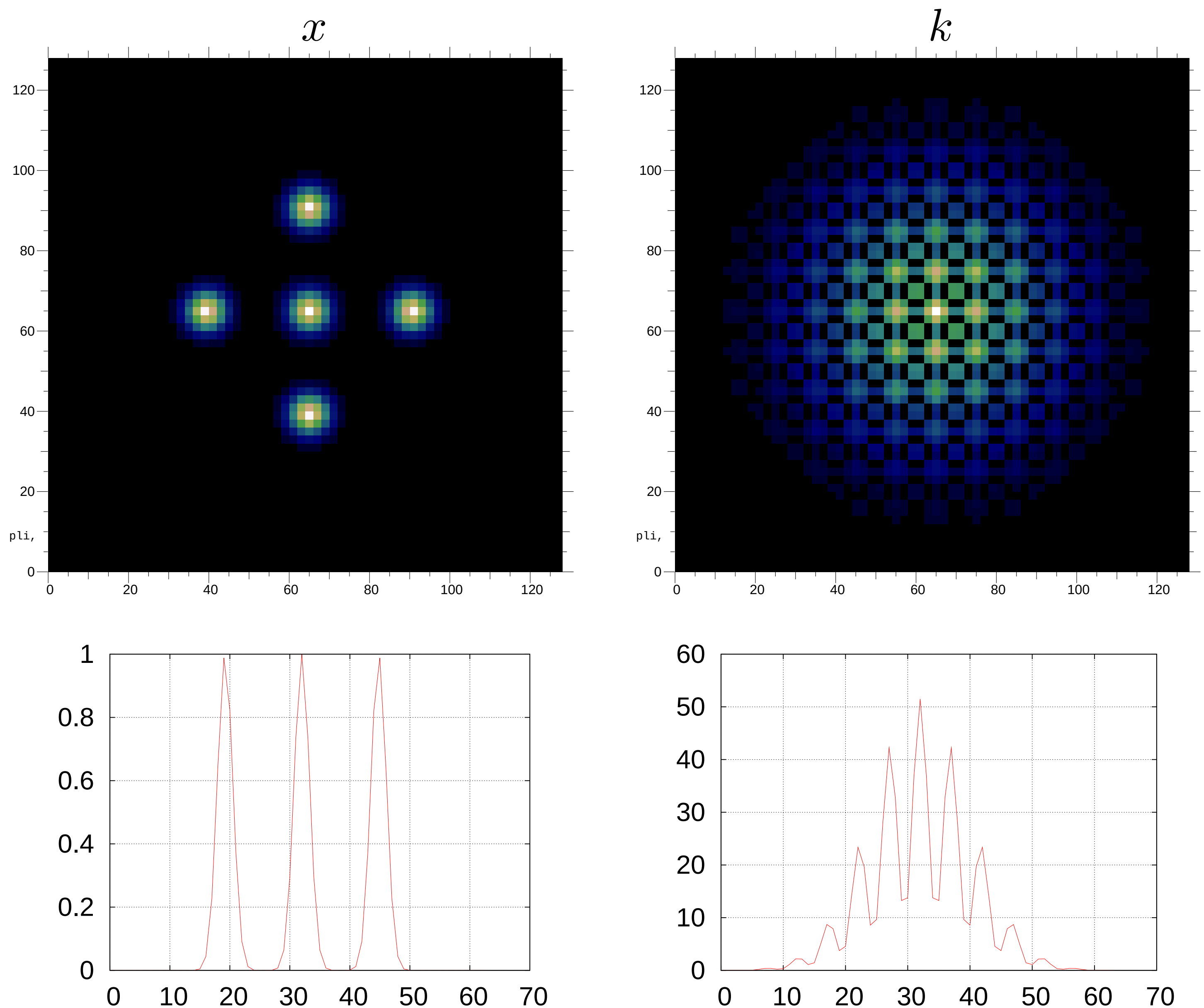








Blobs in Fourier space



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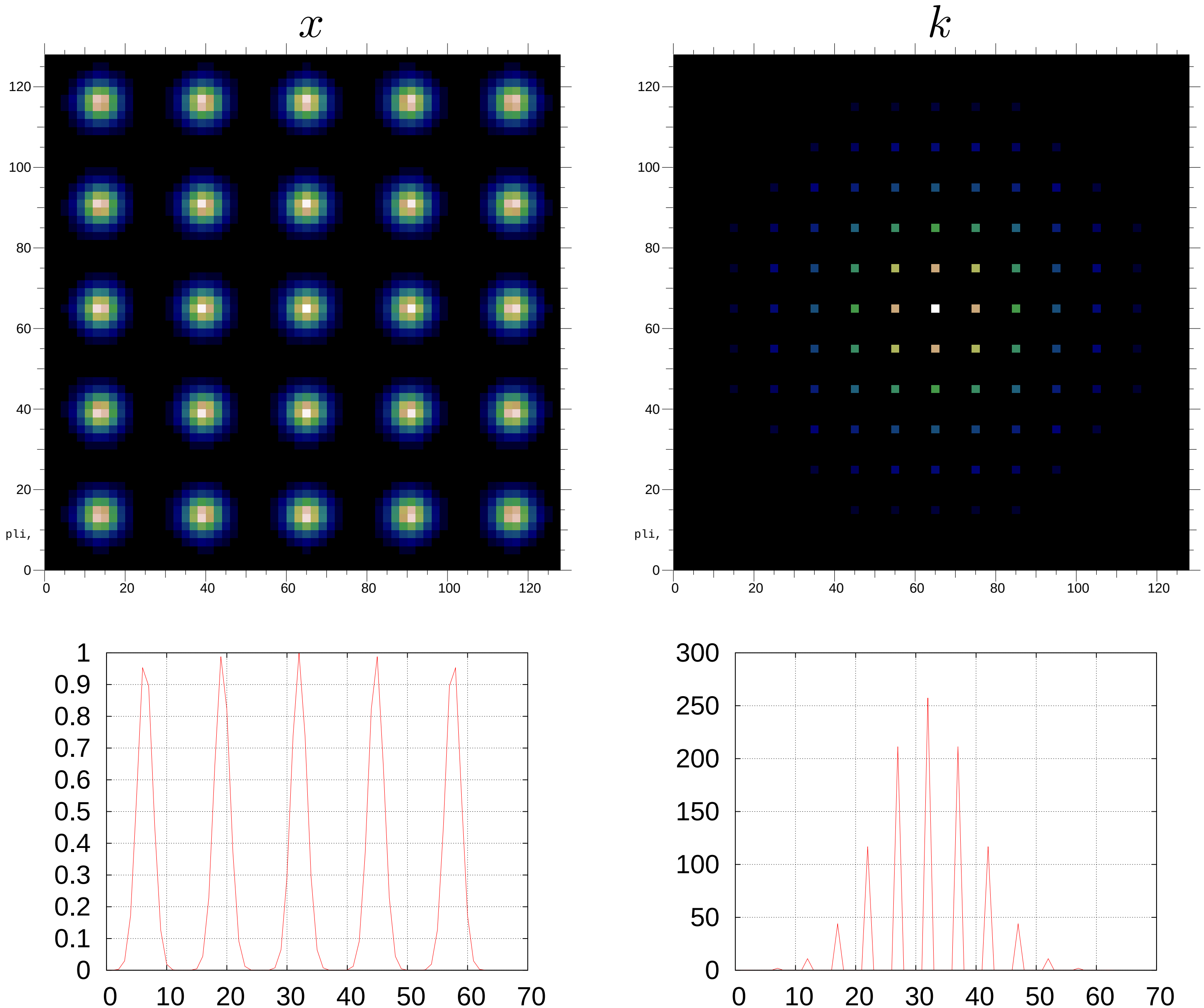

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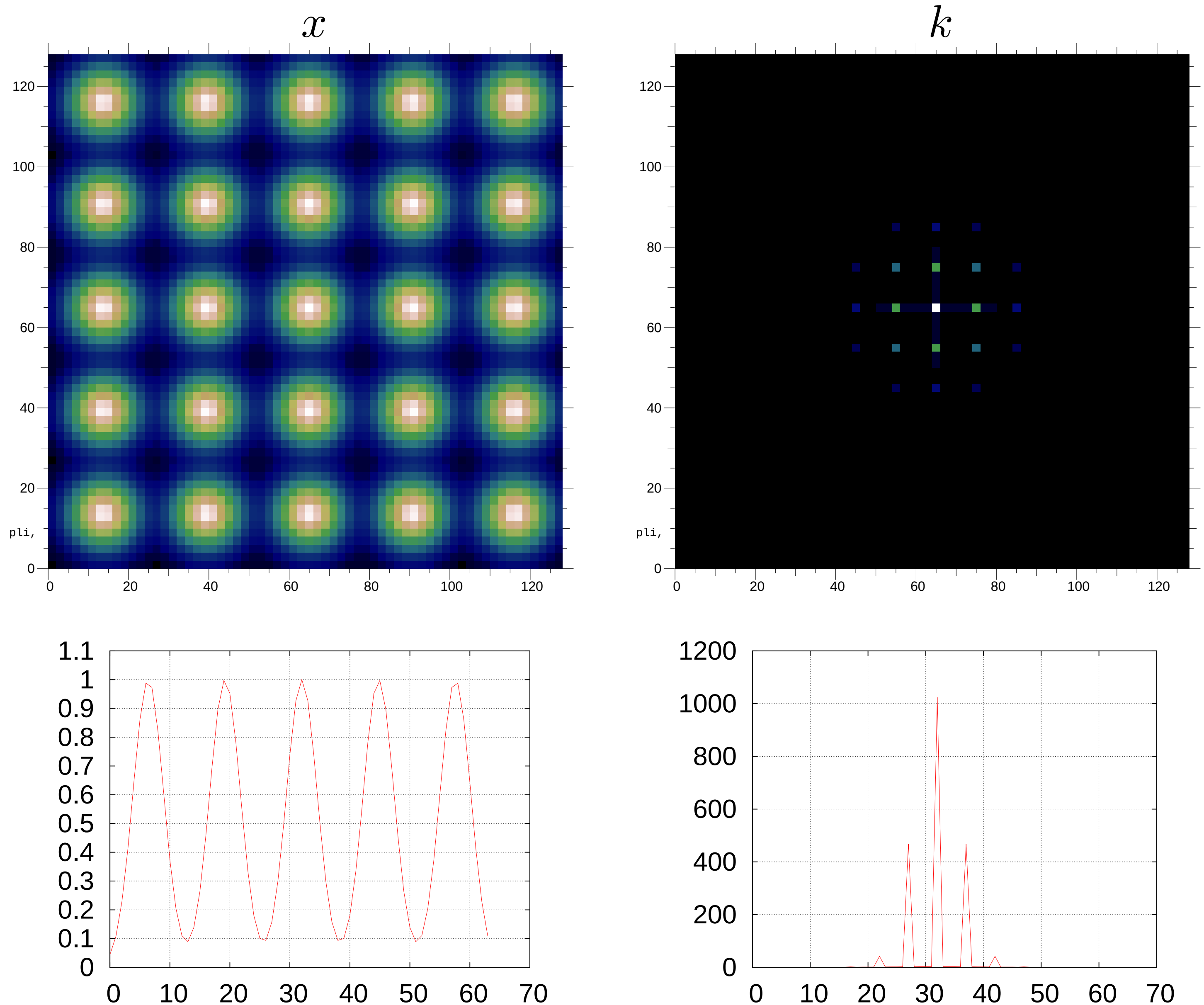

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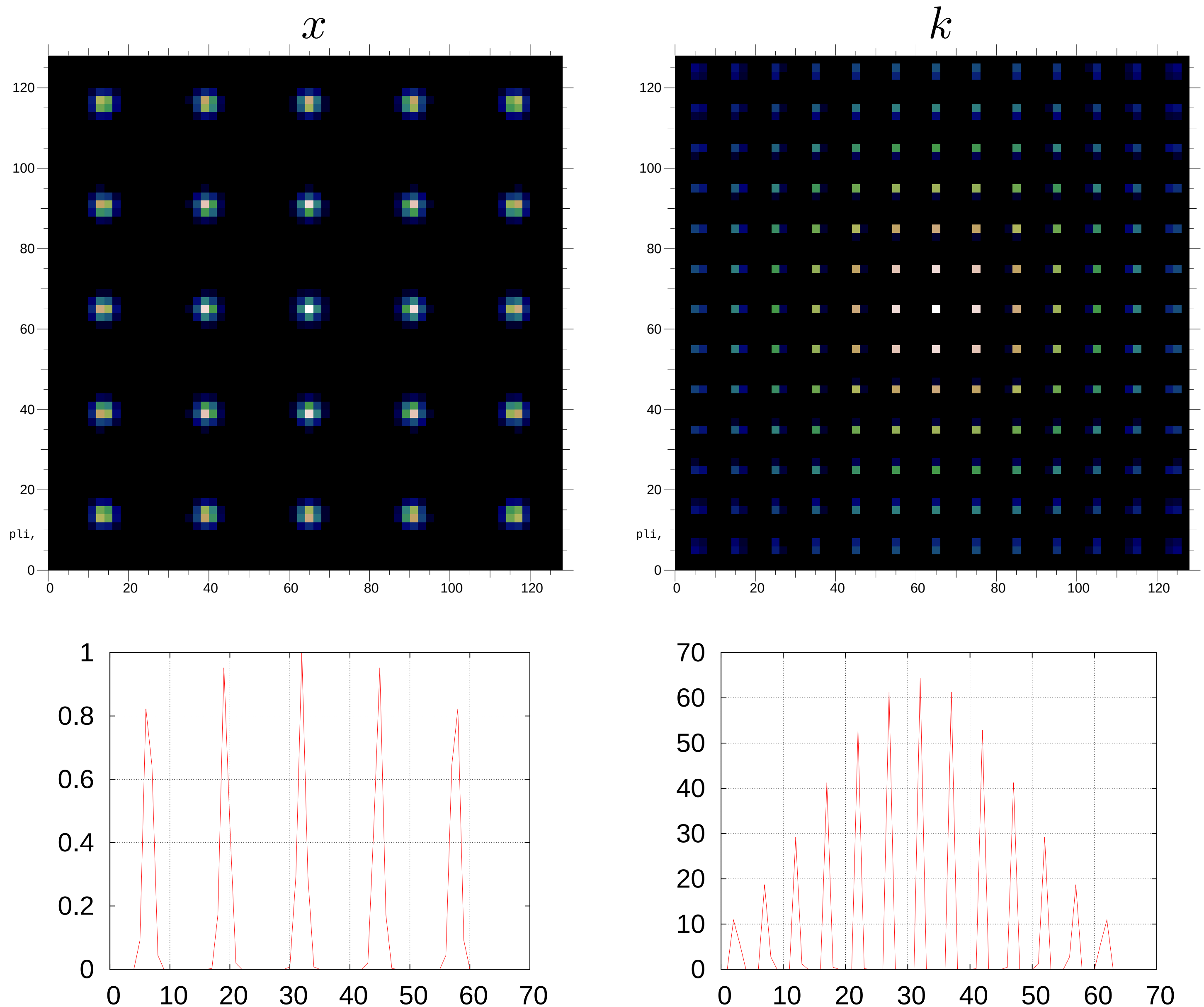

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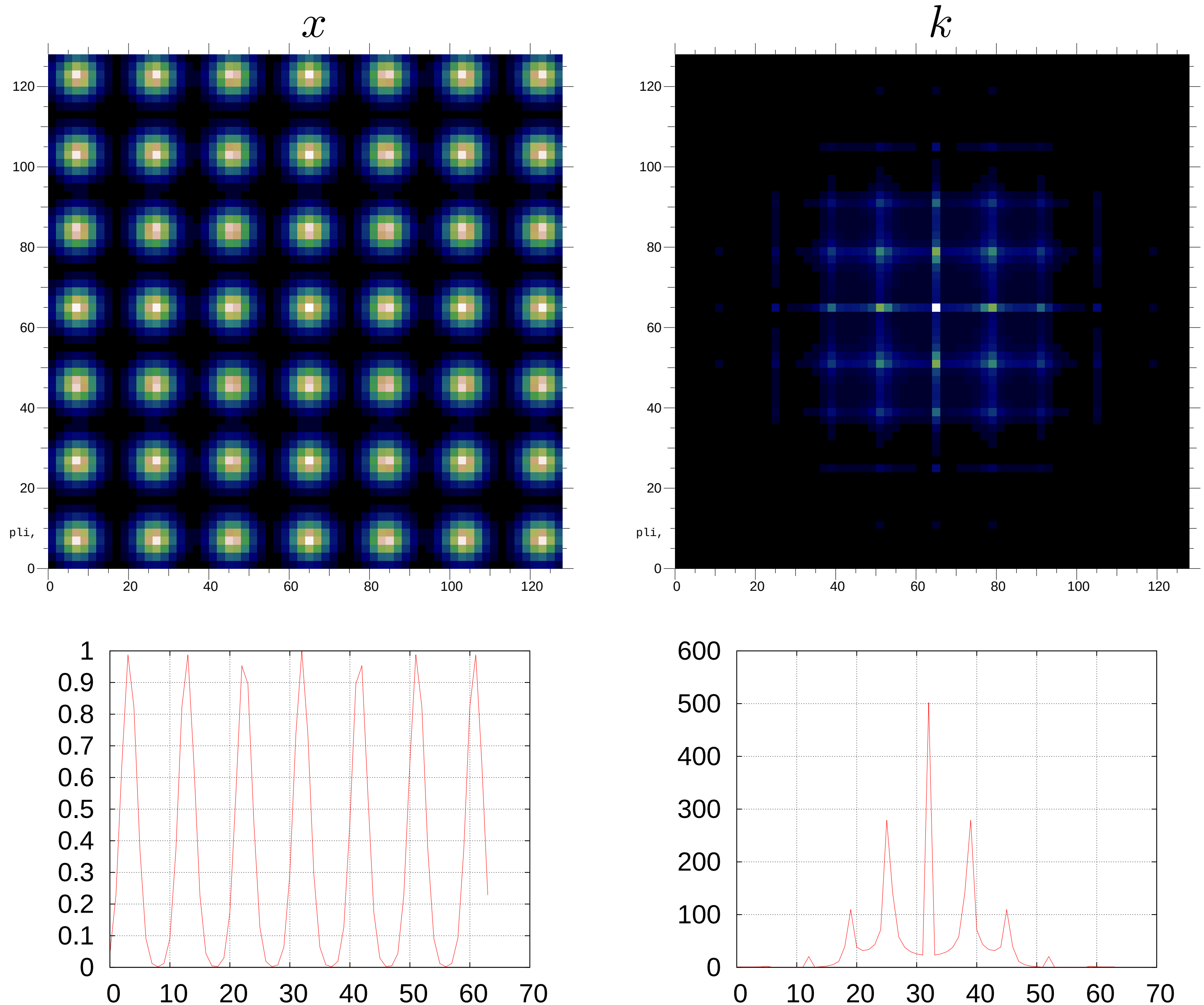


Linear operators

General linear operator:

$$Lf(x) = \sum_y K(x,y)f(y)$$

Translation-invariant:

$$Lf(x) = \sum_y K(x-y)f(y) = \sum_y K(d)f(x-d)$$

0	0	0	0	0
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
0	0	0	0	0

averaging

0	0	0	0	0
0	0	0	0	0
0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0
0	0	0	0	0
0	0	0	0	0

derivative

0	0	0	0	0
0	0	1	0	0
0	1	-4	1	0
0	0	1	0	0
0	0	0	0	0

Laplacian

Convolutions

Convolution:

$$(f * g)(x) = \sum_y f^*(x-y)g(y) = \sum_y f^*(y)g(x-y)$$

Template matching:

- $g(x)$ is template function, centered around $x = 0$
- $(f * g)(x)$ gives *overlap* of f, when g is f is translated to x

Smoothing:

- $g(x)$ is smoothing kernel, centered around $x = 0$
- $(f * g)(x)$ is $f(x)$ with each value replaced by a local average

Convolutions in Fourier space

$$\begin{aligned}
\mathcal{F}(f*g)(k) &= \sum_x (f*g)(x)\, e^{-ik\cdot x} \\
&= \sum_x \sum_y f(x-y)\, g^*(y)\, e^{-ik\cdot x} \\
&= \sum_x \sum_y \sum_{k_1,k_2} \hat{f}(k_1) e^{ik_1\cdot(x-y)}\, \hat{g}^*(k_2) e^{-ik_2\cdot y}\, e^{-ik\cdot x} \\
&= \sum_{k_1,k_2} \hat{f}(k_1)\, \hat{g}^*(k_2) \sum_{x,y} e^{i(k_1-k)\cdot x} e^{-i(k_2-k)\cdot y} \\
&= \hat{f}^*(-k)\, \hat{g}(-k)
\end{aligned}$$

using orthogonality:

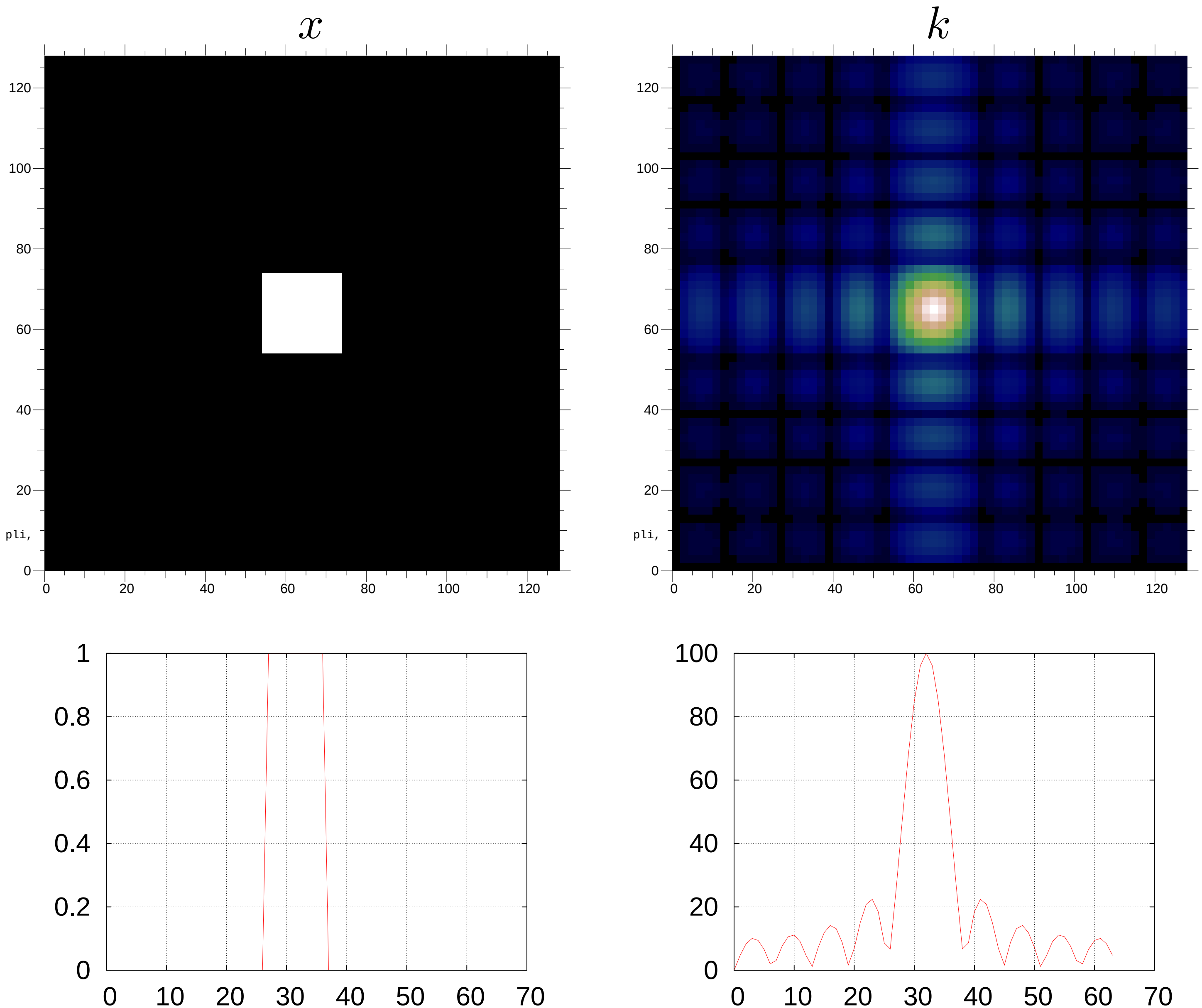
$$\sum_x e^{ik\cdot(x-y)} = \delta_{x,y}$$

Convolutions in Fourier space

$$\mathcal{F}(f * g)(k) = \hat{f}^*(-k)\,\hat{g}(-k)$$

- Convolution in real space is multiplication in Fourier space
- Fast algorithm for
  - Filtering
  - Matching
  - Numerical operations

Convolution in Fourier space



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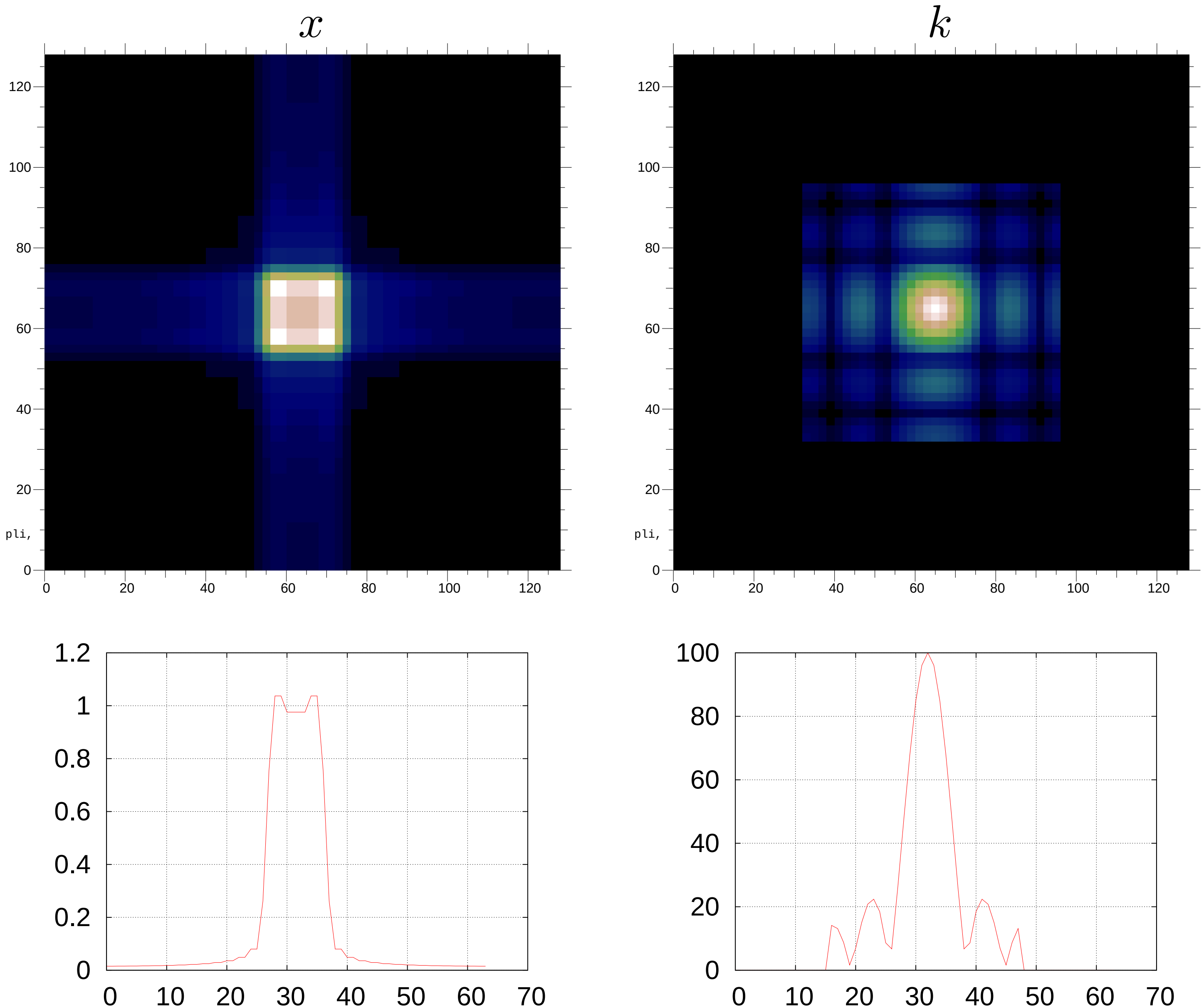
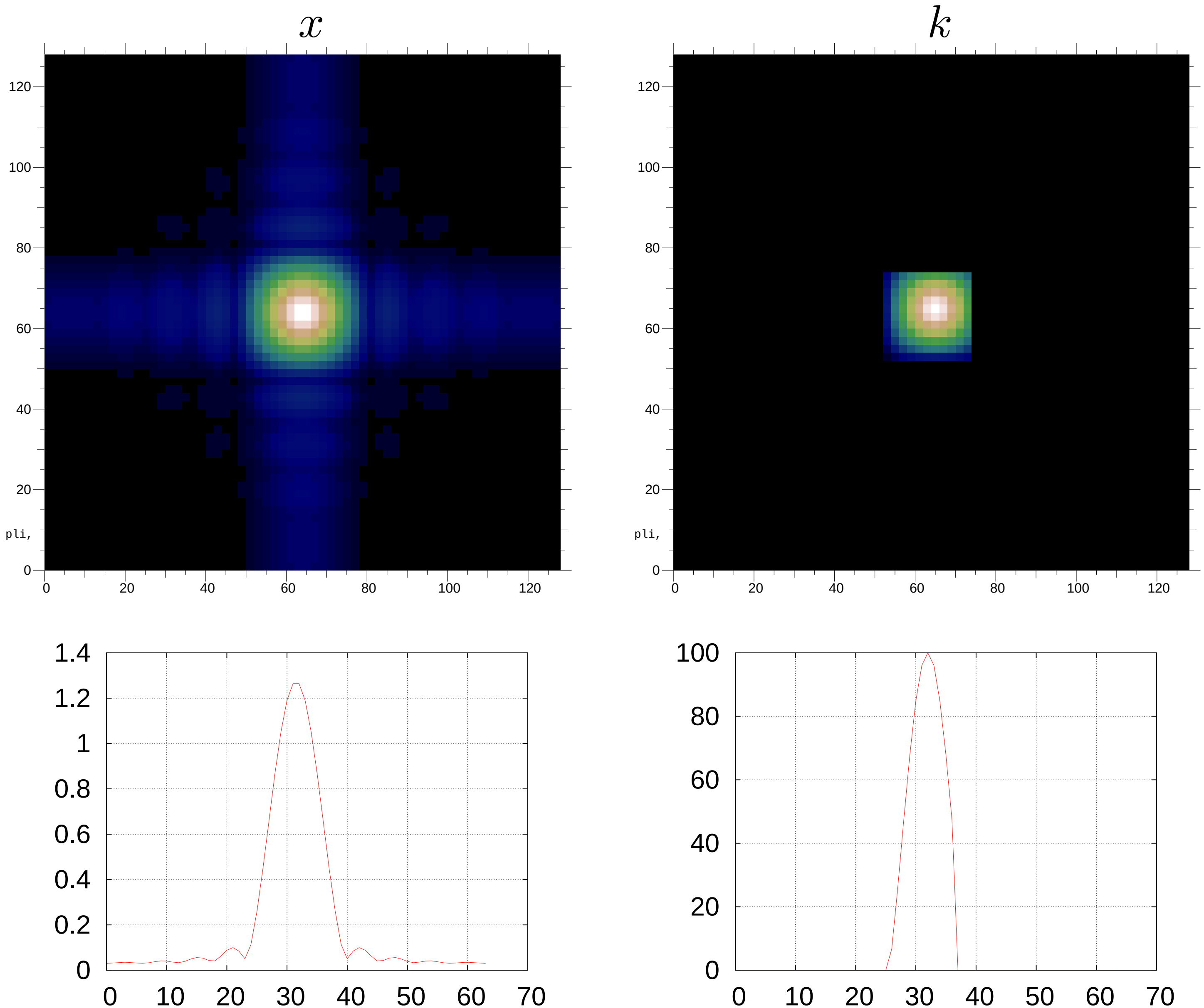
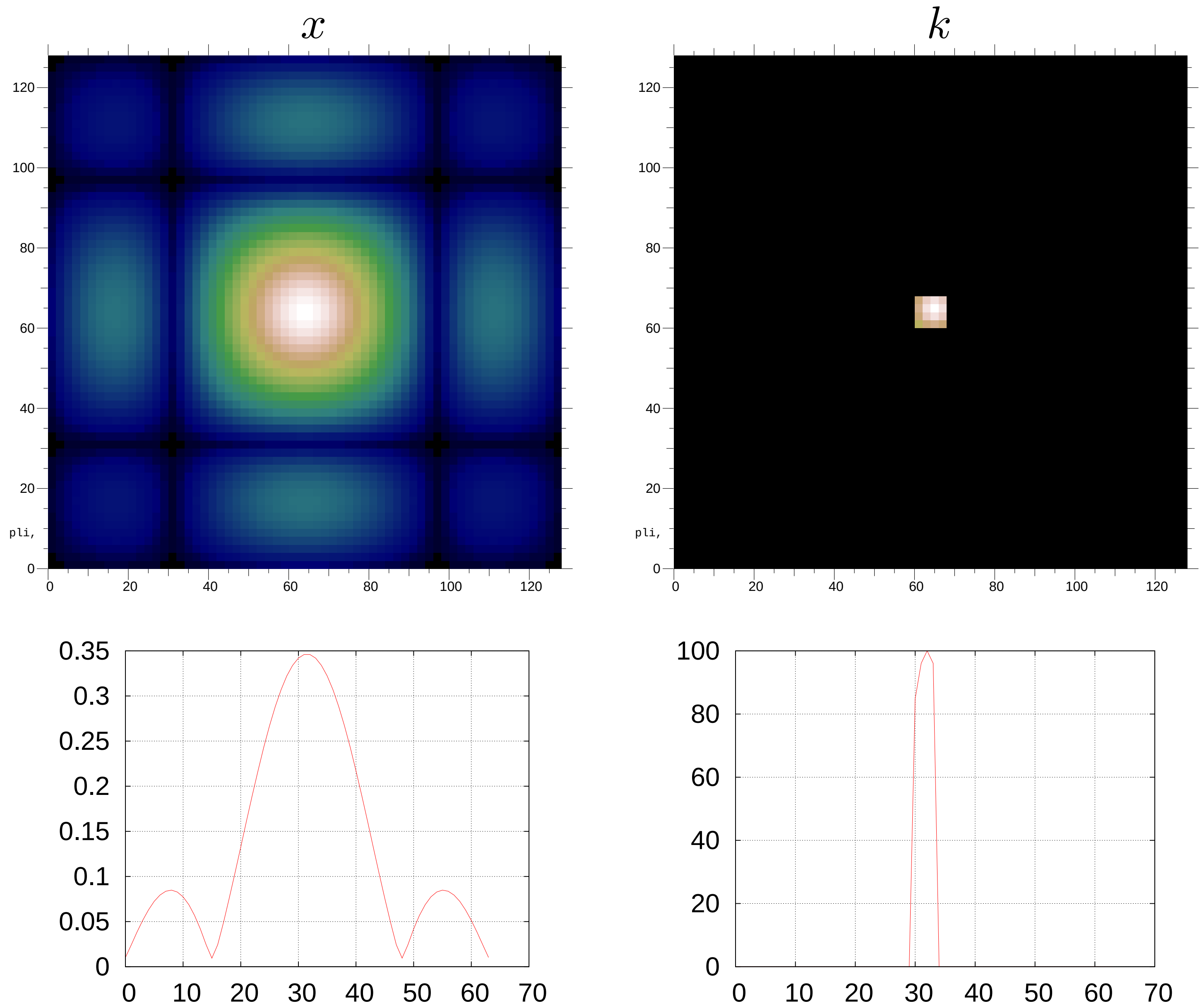


Convolution in Fourier space



Convolution in Fourier space



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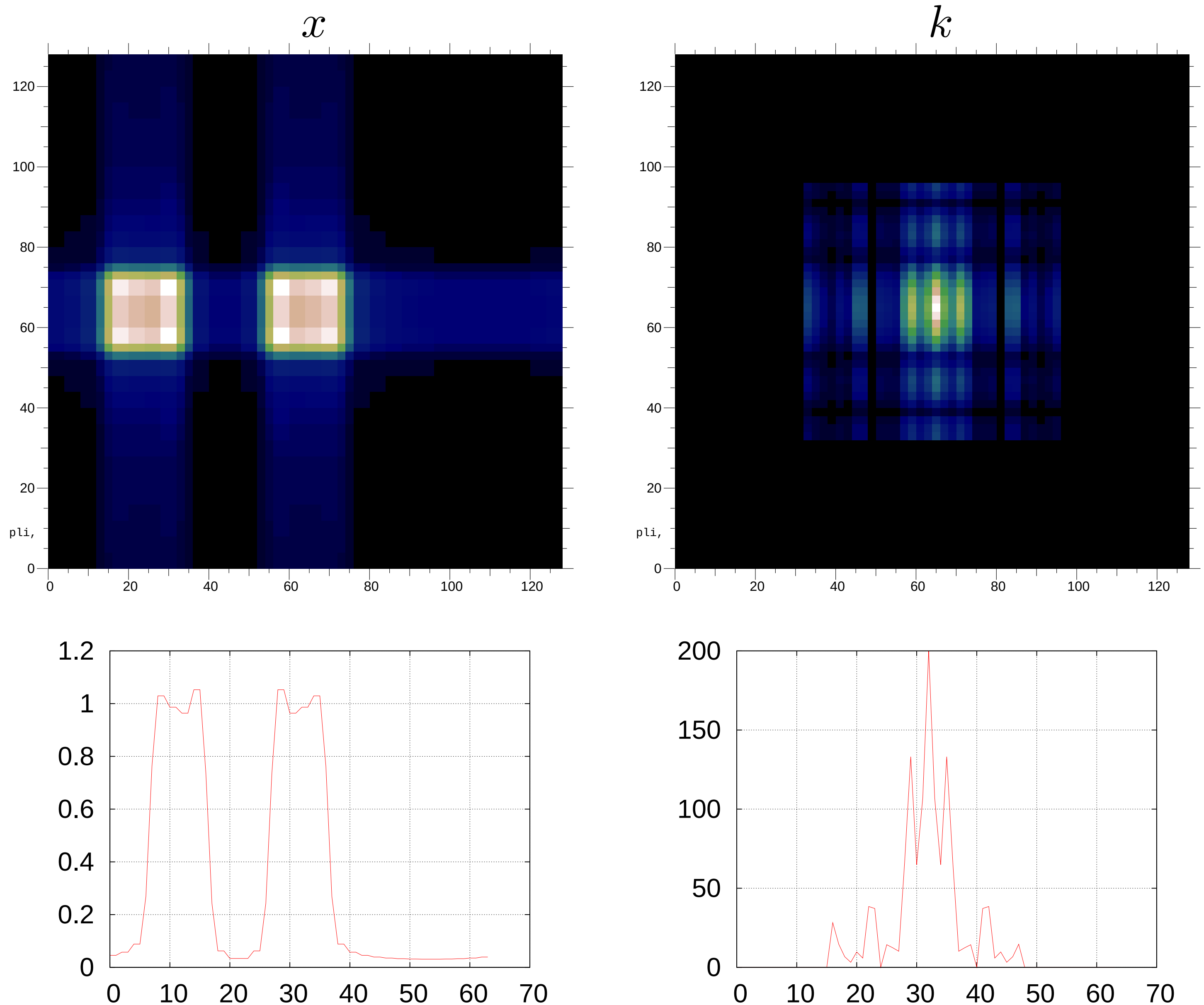
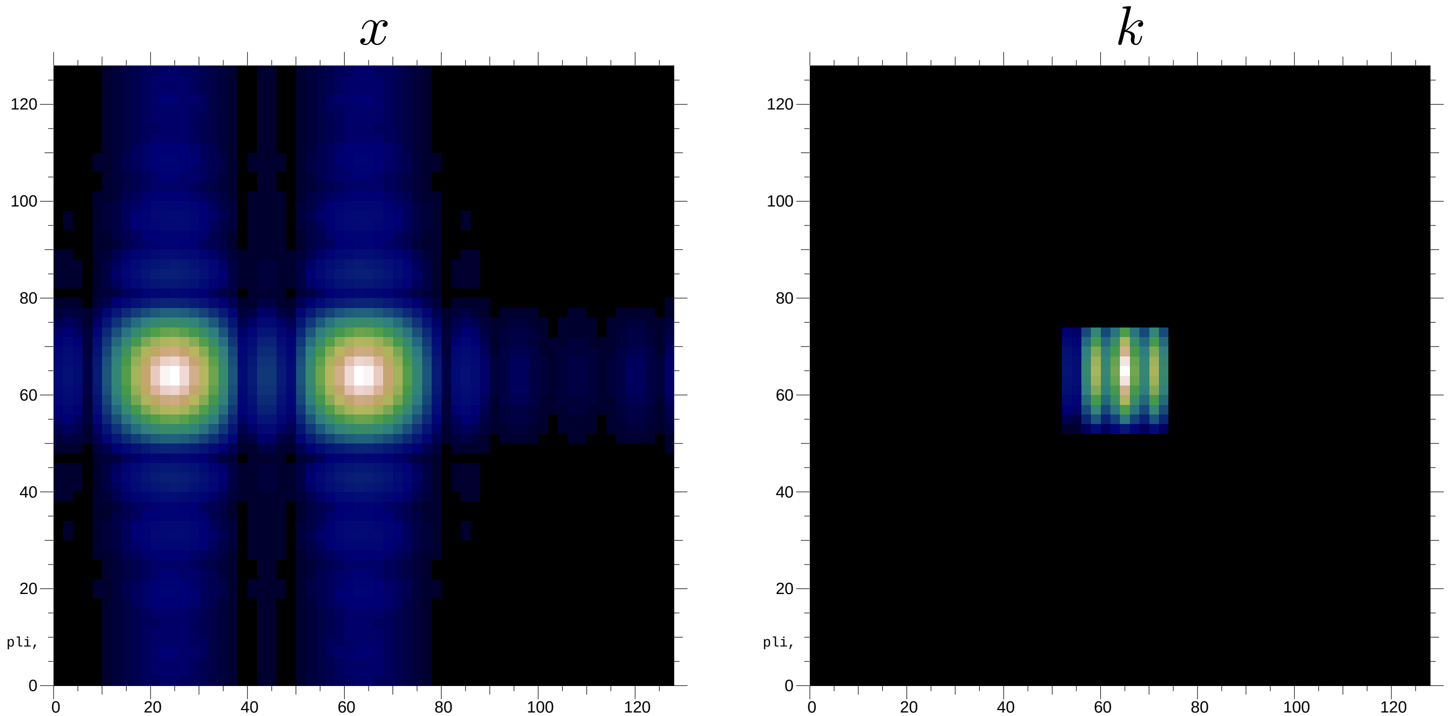
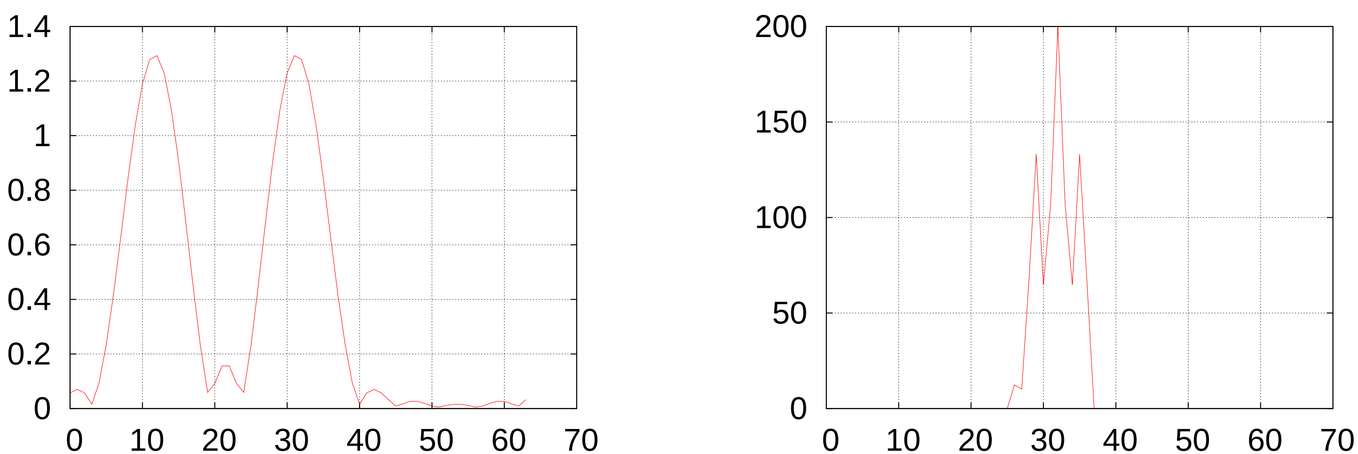
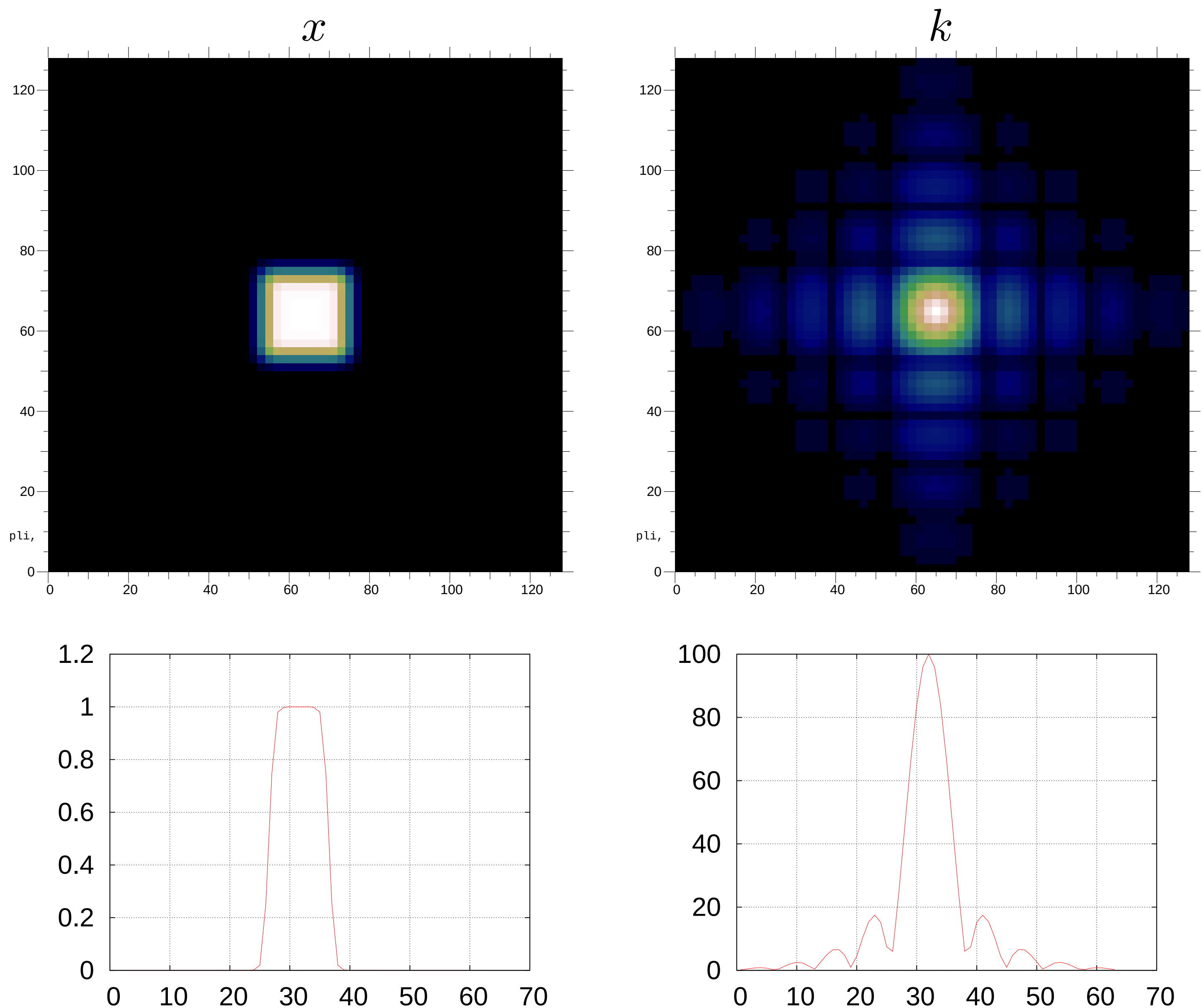


Convolution in Fourier space





Convolution in Fourier space



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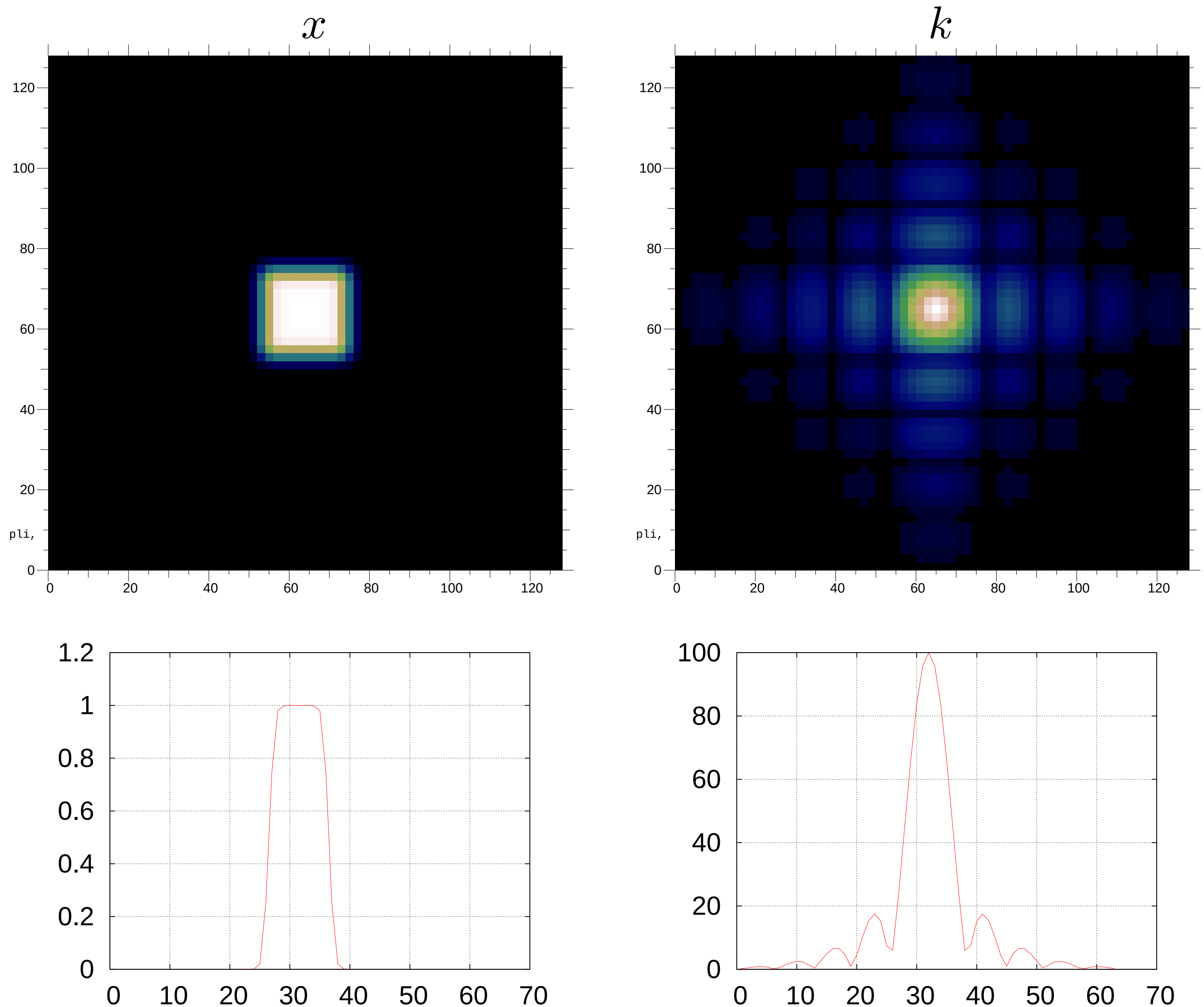

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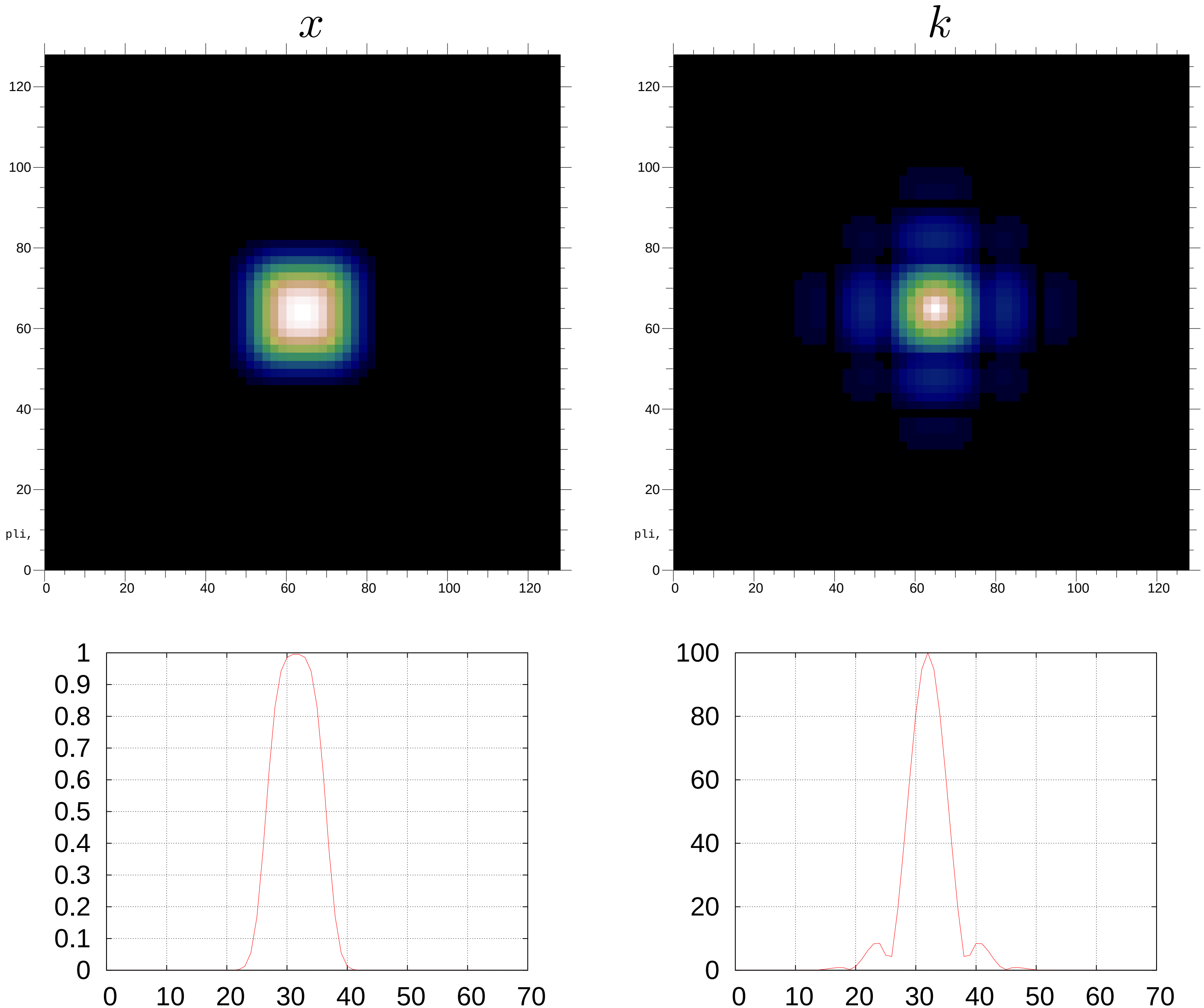
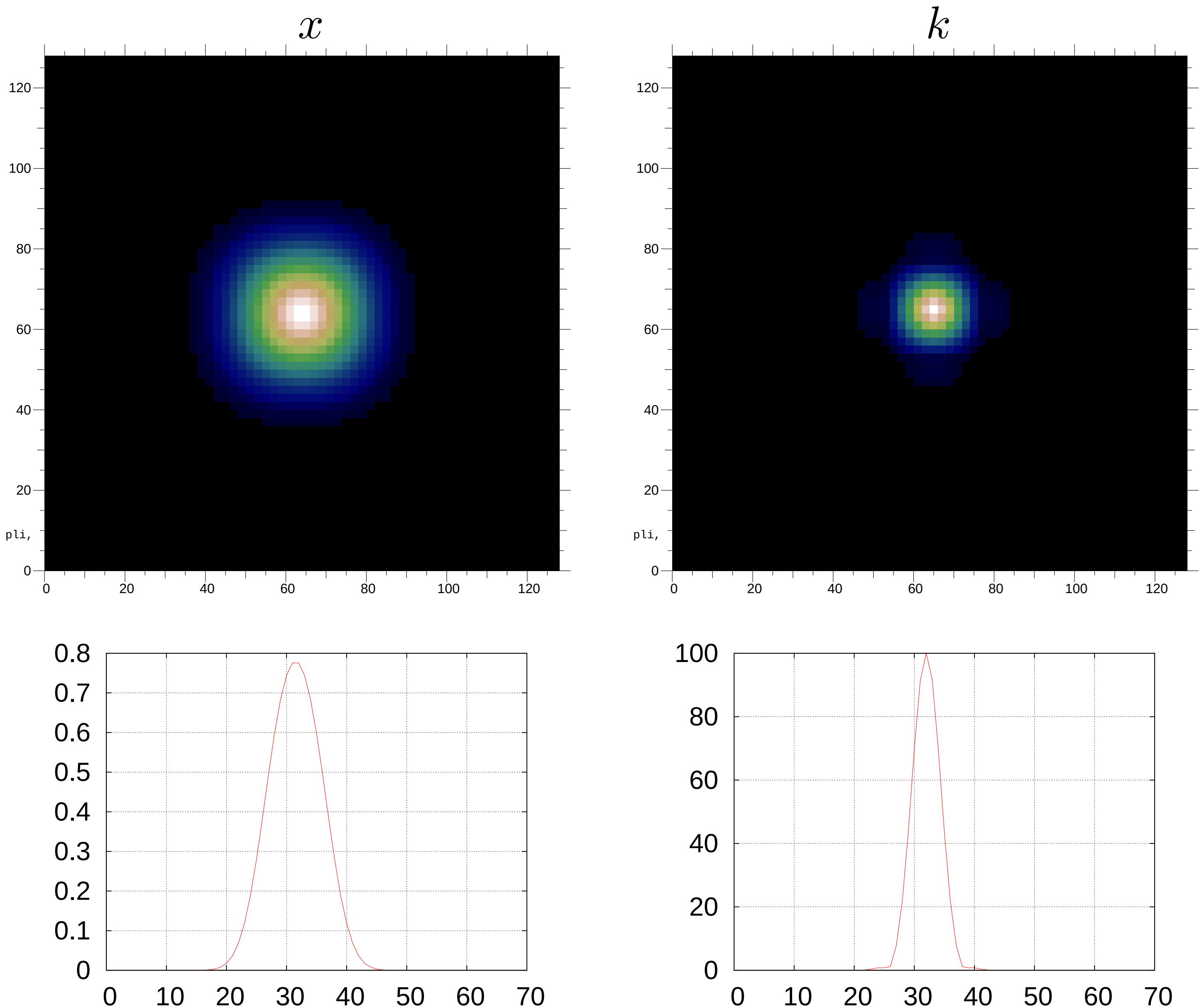


Convolution in Fourier space



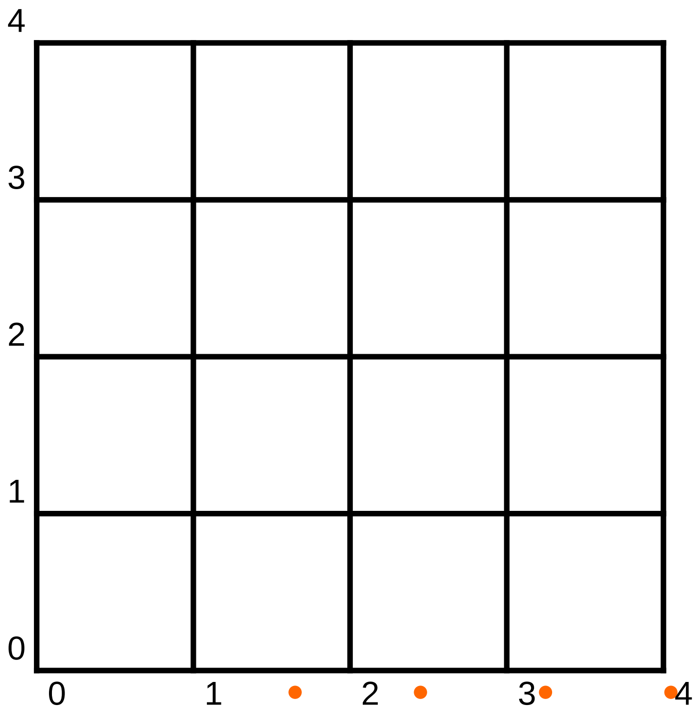
Aliasing

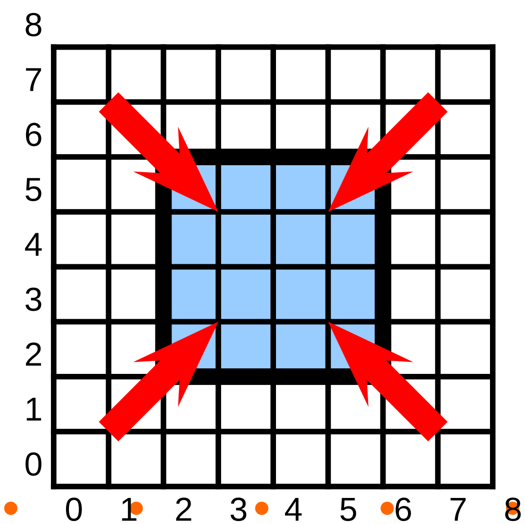
Resampling: Leave out every other pixel

$$\Delta x \longrightarrow 2\Delta x \qquad k_{\max} \longrightarrow \frac{k_{\max}}{2}$$

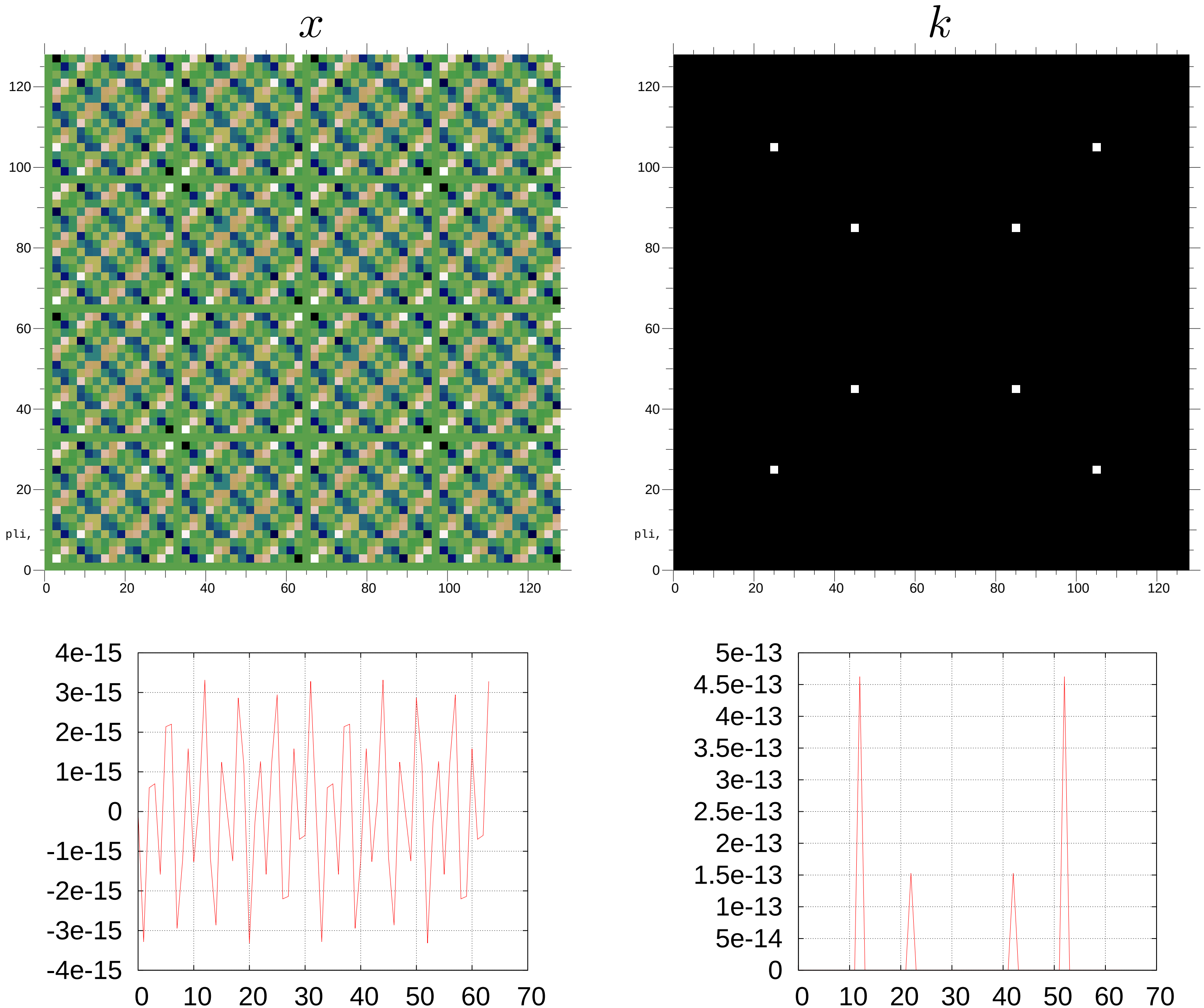
In Fourier space

$$\hat{g}(k) \quad = \quad \hat{f}(k') + \hat{f}\left(k' + \frac{\pi}{\Delta x}\right)$$

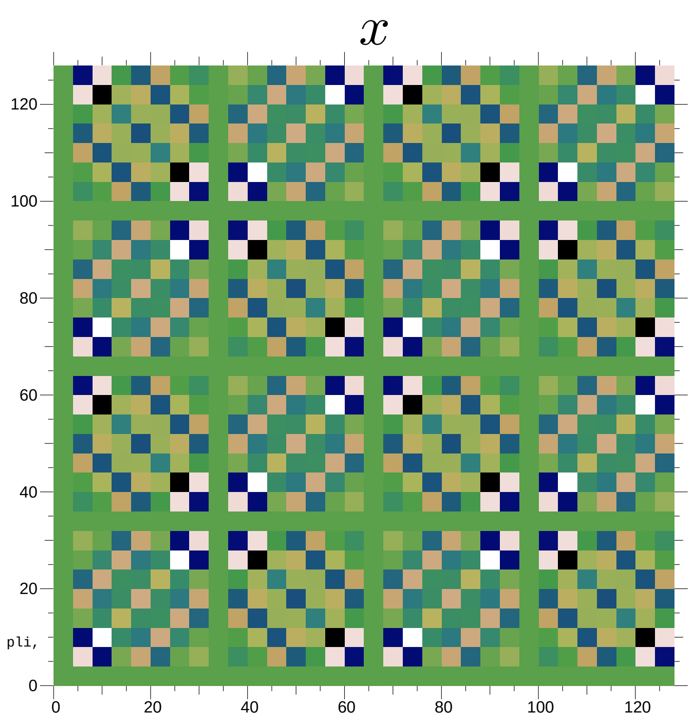


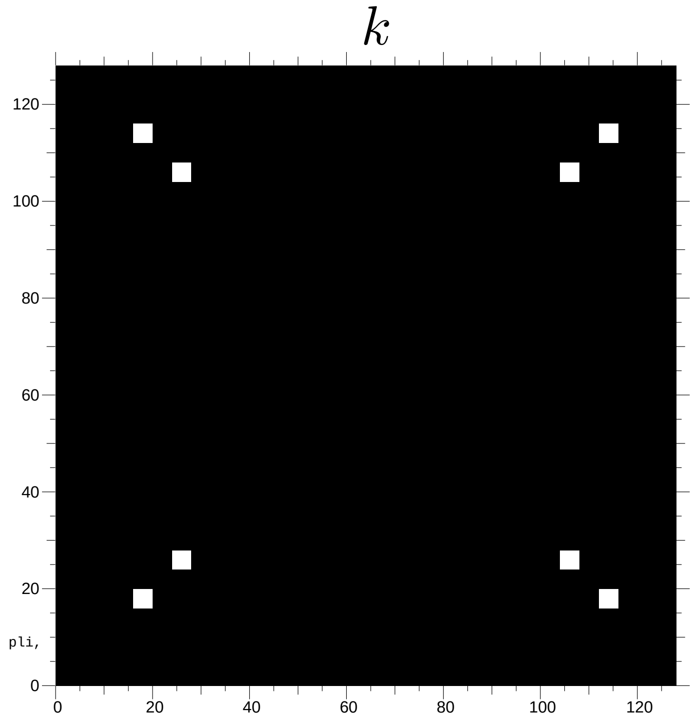


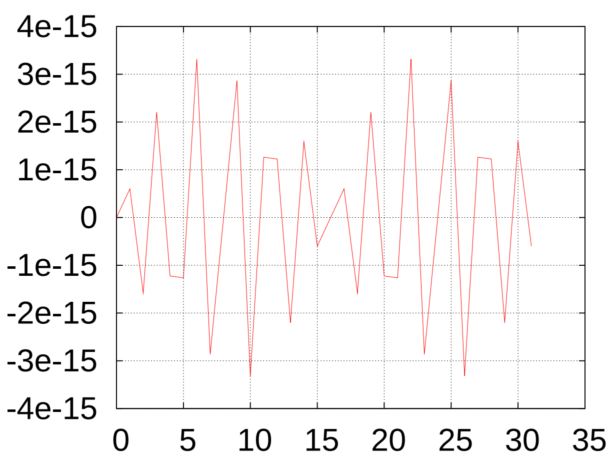
Aliasing: high-resolution

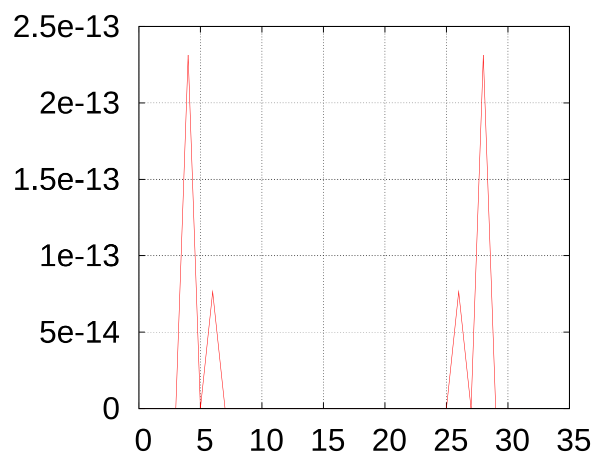


Aliasing: downsampled

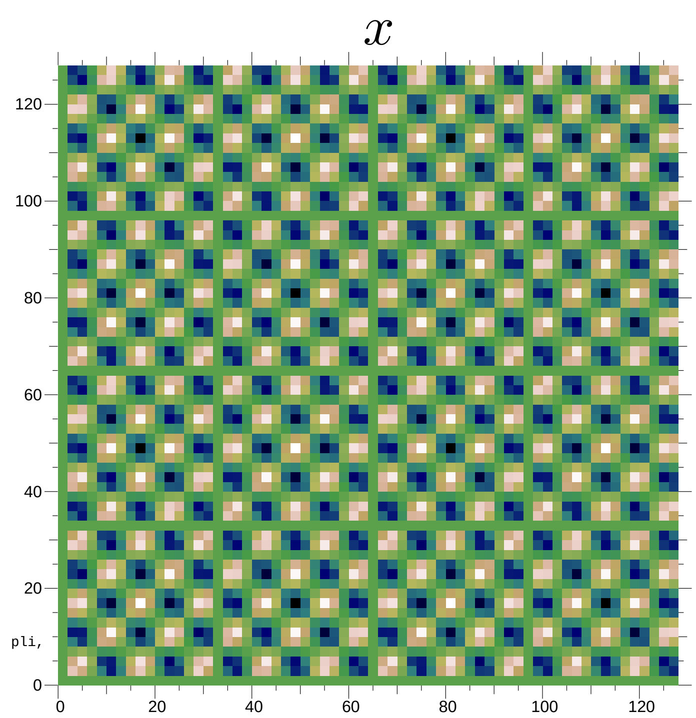


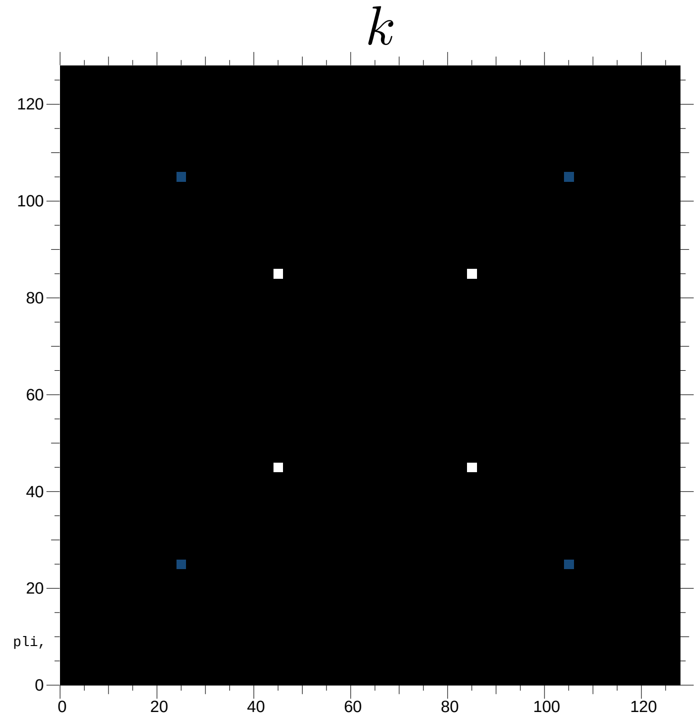


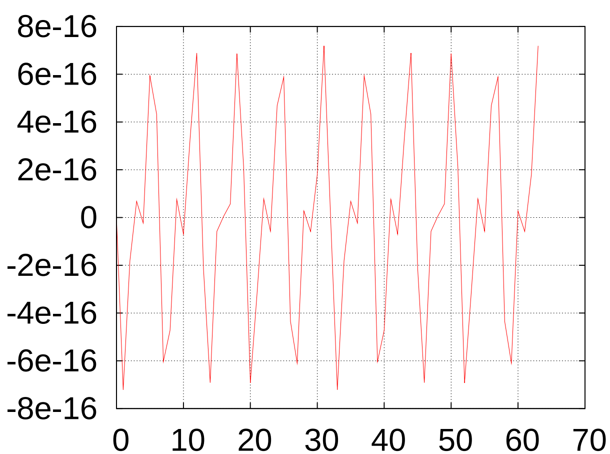


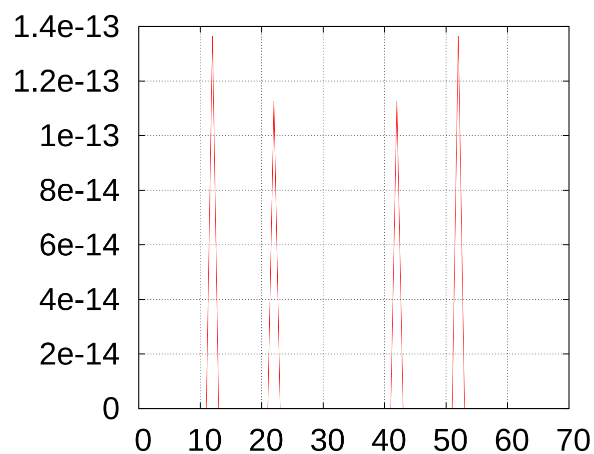


Aliasing: filtered









Aliasing: filtered and downsampled